COURANT AND HILBERT ON MATHEMATICAL PHYSICS

Methoden der mathematischen Physik. Bd. I. By R. Courant and D. Hilbert. Berlin, Julius Springer, 1924. xiii + 450 pp.

The book under review is the twelfth volume of the series *Die Grundlagen der mathematischen Wissenschaften in Einzeldarstellungen.* It is the second book of this series to be devoted to mathematical physics, it being preceded by volume IV, Madelung's *Die mathematischen Hilfsmittel des Physikers.** This earlier volume covers very extensive ground but, necessarily, in a rather cursory manner. The present volume, on the contrary, centers around one single physical problem, the oscillation problem, with its mathematical equivalents, the boundary value and expansion problems.

These are the main problems. Incidentally the reader will pick up a good deal about methods which are applicable to other problems of mathematical physics, but he will have to supply the applications himself. However, in these days of *Morbus relativitus* the information might be welcome that the word tensor appears on page 3 of the book and disappears on page 30, and it is not frequently used.

A few words regarding the joint authorship should be appropriate. The book is obviously and avowedly written by Courant. It is true that most of the subject matter originated directly or indirectly with Hilbert, whose spirit hovers over almost every page of the book. But the reader can easily verify, by looking up the several references, that a considerable portion of the book is based upon Courant's own investigations. This is especially the case with Chapter VI. Otherwise, the simple choice of methods, the fondness of heuristic considerations and a certain delicate touch of the pen, sometimes a bit vague but always elegant, betray the writer if nothing else does. All these qualities make the book easy and enjoyable reading.

We have already mentioned that the book deals with the oscillation problems of mathematical physics. This theory culminates in Chapters V and VI of the book, the former giving the equivalent boundary value and expansion problems, the latter the properties of the characteristic values and functions. The existence of the solutions is, occasionally postulated, many existence proofs being postponed to the second volume of the book which will appear later. The first four chapters lay a foundation for the theory; they deal with linear transformations and quadratic forms, expansions in terms of orthogonal functions, linear

* Review by B. O. Koopman, this BULLETIN, vol. 30, Nos. 5–6 (May–June, 1924), p. 272.

integral equations and the elements of the calculus of variations. In the seventh and last chapter some special functions defined by boundary value problems are dealt with in more detail.

The methods employed in the book are mostly very simple. Equicontinuity and Bessel's inequality are the only fairly novel tools which are used extensively. The integrals occurring are seldom interpreted in the sense of Lebesgue. Stieltjes and Hellinger integrals are completely avoided. In order to be able to place themselves on such an elementary basis the authors purposely restrict the discussion to functions which, together with the necessary number of derivatives, are continuous in adjacent intervals (stückweise stetig).

The leading idea in the greater part of the book is that the characteristic values and the characteristic functions* which belong to a given oscillation problem, satisfy certain *maximum-minimum conditions.* Thus the nth characteristic value is the upper limit for the minimum of a particular quadratic functional $F(\varphi)$ associated with the problem when the function φ is properly normalized and satisfies $(n-1)$ variable conditions of orthogonality. The function φ_n, supposed to exist, for which $\operatorname{Min} F(\varphi)$ reaches its upper limit, is the nth characteristic function; the corresponding $(n-1)$ orthogonal functions are the preceding characteristic functions. This *independent* definition of the nth characteristic value is due to Courant, though it seems to have been expressed for finite quadratic forms earlier by E. Fischer; the classical *recurrent* definition goes back to H. Weber. This simple principle is manipulated with great dexterity and yields surprisingly rich results.

The reader meets this guiding notion as early as on page 11 in the first chapter where the principal axes of a central hyper-quadric are found. It is clearly expressed for quadratic forms on page 17. The whole first chapter is essentially an algebraic analog from the theory of quadratic forms to the oscillation theory for continuous bodies developed later. It is the well known Hilbert theory oriented and adapted for a particular purpose. The second chapter opens with a general discussion of orthogonal functions, followed by a discourse on equicontinuous functions. The reader should notice the correction on page xiii referring to the discussion on page 40. In § 3 we meet the notions of *measure of independence m* and *asymptotic dimension number r* for a set of functions $f_i(x)$. The former is defined as the minimum of the quadratic form

$$\sum_{i,j=1}^{n} f_{ij} t_i t_j \quad \text{for} \quad \sum_{i=1}^{n} t_i^2 = 1, \quad \text{where} \quad f_{ij} = \int_a^b f_i f_j \, dx.$$

* Eigenwerte und Eigenfunktionen. The reader should notice that the authors use the term characteristic numbers (charakteristische Zahlen) for the reciprocals of the characteristic values.

If the functions $f_1, \cdots, f_n$ are linearly dependent, $m = 0$; if they are normalized and orthogonal, $m = 1$. The authors do not mention that this is the maximum value of m for normalized functions. Similarly the Gramian is discussed at various places, but the authors fail to notice that $\det(f_{ij}) \leqq f_{11}f_{22} \cdots f_{nn}$, where the equality sign holds if and only if the functions are orthogonal to each other.* The asymptotic dimension number is defined as the least integer r such that, if $s > r$, the measure of independence of the functions $f_{n_1}, f_{n_2}, \cdots, f_{n_s}$ converges to zero when $n_1, n_2, \cdots, n_s \to \infty$. The remainder of the second chapter is devoted to special orthogonal systems and to the approximation theorem of Weierstrass, Fejér's proof for trigonometric polynomials, and Landau's for ordinary ones.

In the third chapter we find an outline of the theory of integral equations. In addition to the classical theory of such equations, we find two new methods for the existence proofs. Both are based upon uniform approximation of the kernel by degenerate kernels, but the method of extracting a convergent sequence from the approximate solutions differ. One method employs the properties of equi-continuous functions, the other the asymptotic dimension number. For the symmetric case, the maximum-minimum properties of the characteristic values and functions play a fundamental role in the discussion. We miss a reference to Heywood and Fréchet's treatise in the bibliography appended to the third chapter.

The introduction is the most interesting part of Chapter IV, the elements of the calculus of variations. Here we are granted a foretaste of the direct methods of solving variational problems by means of minimal sequences which subject will be one of the main features of the second volume. In § 9 Hamilton's principle is introduced, with the aid of which the differential equations of vibrating masses are derived. The other parts of the chapter are more useful than exciting.

These differential equations form the main object of Chapter V. Here the boundary value problem is reduced to the solving of a symmetric integral equation by means of the corresponding Green's function. The existence of such a function is proved merely in the linear case; for two or more dimensions it is postulated and the proof will be given in the second volume. A solution of the expansion problem is given which is improved upon in the following chapter by reducing the restrictions. Various simple boundary value problems are treated explicitly.

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* For a proof see O. Dunkel, *Integral equalities with applications to the calculus of variations*, American Mathematical Monthly, vol. 31 (1924), pp. 326–337. This interesting paper unfortunately does not give any references to the literature.

In Chapter VI we reach the climax. The maximum-minimum principle is given full play and is used for a very interesting qualitative discussion of the dependence of the characteristic values upon the data of the problem, namely the coefficients of the equation, the basic region and the conditions on the boundary. This discussion is facilitated by the following simple observation: The minimum of the functional $F(\varphi)$, when φ ranges over a certain field of functions, is not decreased by restricting the field, and not increased by enlarging the field. The same principle gives a simple determination of the asymptotic distribution of the characteristic values, the basic region under consideration being approximated by rectangles for which the characteristic values are known. An application to the black-body problem is given.

The seventh chapter is not so well done as the rest of the book. To be sure, there are brilliant points: the interpretation of Laplace integrals at the beginning of the chapter and the presentation of the saddle-point method at the end are gems. But the long and involved treatment of Bessel's functions seems scarcely justifiable except on the basis that contour-integration is a method of mathematical physics. It would have been simpler to start directly with the differential equation for $J_\lambda(x)/x^\lambda$ on page 399 and leave out most of the preceding discussion for which the reader could have been referred to the corresponding parts of Watson's standard treatise. A graver remark must be directed against the discussion of the zeros of Bessel's functions on pages 412–414. The formula (33) on page 413 becomes illusory for $\lambda < -1$, and the conclusions drawn in this case concerning the complex zeros of $J_\lambda(x)$ are erroneous. On page 419 Legendre's function of the second kind, $Q_\nu(x)$, is defined by an integral for $\Re(\nu) > -1$ and by the relation $Q_\nu(x) = Q_{-1-\nu}(x)$ for $\Re(\nu) < 0$. This implies a contradiction in the common strip which could be avoided by making $\Re(\nu) = -\frac{1}{2}$ the dividing line between the two definitions. Why should the symbol $Q_\nu(x)$ represent different analytic functions of ν in different parts of the ν-plane? Why not use the original definition by a contour-integral which is valid for every value of ν, not a negative integer?

This finishes our survey of the different chapters. If our labor has not been in vain, it ought to be clear to the reader of this review that the book, in spite of its restricted scope, is rich in material and in points of view which are either novel or little known. The book — as most human work — is not perfect, but the imperfections are mostly on side-issues. It was obviously not meant as an opiate, but intended to stimulate interest, discussion and research in a field which still belongs to the richest in mathematical physics. We look forward to the appearance of Volume II with eager expectation.

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