THE BOREL THEOREM AND ITS GENERALIZATIONS*

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That any one should attempt to devote a paper to the subject of the Borel Theorem may at first glance seem a presumption. A brief investigation will however reveal the following facts. (a) The Borel Theorem is closely related to the fundamental postulates of linear order. (b) There are many extensions and analogs of the Borel Theorem, some of which are hidden away in papers on other subjects. (c) The Borel Theorem has held and still holds a central position in the development and analysis of general spaces.

The arrangement of topics in the paper is suggested by the previous paragraph. No claim is made for completeness with respect to the extensions and analogs of the Borel Theorem, due to the nature of the case. Nor do I claim any originality in the material presented. I hope that a systematic treatment of the Borel Theorem in general spaces will be suggestive and perhaps create further desirable interest and results in these spaces.

I. The Borel Theorem and its Extensions for the Linear Interval and n-Dimensional Space

In order to give a simple statement of the Borel Theorem I shall use the phrase "a family $\mathfrak{F}$ of intervals I covers the point-set E" to mean that every x of E is interior to some interval I_x of $\mathfrak{F}$. Then the Borel Theorem in its simplest form may be stated as follows.

If the family $\mathfrak{F}$ of intervals I covers the closed interval (a, b) then a finite subfamily of $\mathfrak{F}$ covers (a, b).

* An address presented at the request of the program committee before the joint meeting of this Society and the American Association for the Advancement of Science, Section A, at Kansas City, December 30, 1925.

1. *Historical Note.** As in the case of other important mathematical results, the conception of the Borel Theorem is an interesting chapter in mathematical history, to which the future will no doubt add contributions. So far the honor of having first stated an outright theorem on intervals belongs to Borel,† though Borel's formulation deals only with the reduction of a denumerable family of intervals to a finite one. The name Heine-Borel seems to be due to Schoenfliess,‡ who noted the relationship of the Borel Theorem to Heine's proof of the uniform continuity of a function continuous on a closed interval, published in 1872.§ That Heine was aware of the fact that an interval theorem lay hidden away in his proof seems rather doubtful. As a matter of fact priority on the uniform continuity seems to go back at least to G. Lejeune-Dirichlet, though he suffered the penalty of not publishing his result immediately. A proof of the theorem almost identical with that of Heine appears in an exposition of his lectures given in 1854 brought out in 1904 by G. Arendt.|| Another result carrying within it the germs of the Borel Theorem is due to S. Pincherle,¶ who gave the following theorem in 1881.

If the positive-valued function $f(x)$ is bounded from zero in some neighborhood of each point of a closed interval, then there exists an e such that

$$f(x) > e > 0$$

for all points of the interval.

He remarks that this theorem can be made the basis for proof of uniform continuity and of the uniform convergence

* Cf. ENZYKLOPÄDIE DER MATHEMATISCHEN WISSENSCHAFTEN, vol. II_3, pp. 882 et seq.

† Paris thesis, 1894, p. 43; ANNALES DE L'ÉCOLE NORMALE, (3), vol. 12 (1895), p. 51.

‡ *Bericht über die Mengenlehre*, JAHRESBERICHT DER VEREINIGUNG, vol. 8 (1900), pp. 51 and 119. In a later edition (Schoenfliess-Hahn, *Entwickelungen*, vol. I (1913), p. 235), he reverts to the name Borel Theorem.

§ Cf. JOURNAL FÜR MATHEMATIK, vol. 74 (1872), p. 188.

|| *Vorlesungen über die Lehre von den bestimmten Integralen*, Braunschweig, 1904.

¶ MEMORIE DI BOLOGNA, (4), vol. 3 (1881), pp. 151ff.

of a series of functions, uniformly convergent at every point of a closed interval.

To whom shall go the honor of first having conceived of the possibility of extending the Borel theorem to the case where the given family of intervals is not necessarily denumerable is another debatable question. In a way it is true that Dirichlet, Heine, and Pincherle were dealing with this case. Closely related is also the following theorem for the plane, due to P. Cousin.*

If to each point of a closed region there corresponds a circle of finite radius, then the region can be divded into a finite number of subregions such that each subregion is interior to a circle of the given set having its center in the subregion.

Lebesgue, to whom this extension is usually credited, claims to have known the result in 1898,† but first published it in 1904 in his *Leçons sur l'Intregation.* W. H. Young‡ published a proof in 1902. As a matter of fact the statement and proof of the Borel Theorem given by Schoenfliess in his 1900 Bericht can easily be interpreted to be that of the extension in question.

In considering these divergent claims it seems simplest and most just to call the theorem the Borel Theorem, and in case a distinction is necessary indicate it as the denumerable-to-finite or any-to-finite form. Moreover the theorem which gives the reduction from any set of intervals to an equivalent denumerable family due to Lindelöf in a general space proves to be only another case of a more general Borel Theorem.

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* ACTA MATHEMATICA, vol. 19 (1895), p. 22; Fréchet: CONGRÈS SOCIÉTÉS SAVANTES, 1924, p. 68.

† For his attitude on the priority question, see BULLETIN DES SCIENCES MATHÉMATIQUES, (2), vol. 31 (1907), pp. 132–4.

‡ PROCEEDINGS OF THE LONDON SOCIETY, vol. 35 (1902), pp. 384–8.

§ Cf. Schoenfliess-Hahn, *Entwickelungen der Mengenlehre,* vol. 1, 1913, pp. 235 et seq.

|| Cf. Borel (Baire), COMPTES RENDUS, vol. 140 (1905), p. 299; Capelli, NAPOLI RENDICONTI, (3), vol. 15 (1909), p. 151.

2. *Proofs of the Borel Theorem.*§ (a) *By Successive Subdivisions.*|| Of the proofs of the Borel Theorem, perhaps the simplest in form is the one which proceeds by successive subdivisions. It is an indirect proof. Assume that the theorem is not true for the interval (a, b). Then if we divide the interval into two or more equal parts, it will be not true for one of these parts. This process applies indefinitely, and we have a sequence of closed intervals, each containing the succeeding, with lengths converging to zero, giving a single point x of (a, b) common to the intervals. This point being interior to an interval I of the family $\mathfrak{F}$, I will contain the intervals of the sequence after a certain stage, thus yielding a contradiction.

We observe that in a way this proof connects the Borel Theorem with the Cantor Theorem: *An infinite sequence of closed sets of points, each containing the succeeding, have a common point.*

(b) *By Use of the Weierstrass-Bolzano Theorem.* Another method of applying a similar process is to note that if the theorem is not true for (a, b) then if we divide the interval into n equal parts, it will be not true for one of these parts. Let (a_n, b_n) be the interval of length $(1/n)(b-a)$ for which the theorem does not hold, and x_n a point belonging to this interval. Then by the Weierstrass-Bolzano theorem, since the sequence $\{x_n\}$ is bounded, there will exist a subsequence $\{x_{n_m}\}$ having as limit the point x_0 of the interval (a, b). If x_0 is interior to I of the family $\mathfrak{F}$, then, by the properties of limits of sequences, we find that one of the intervals (a_n, b_n) is interior to I.

This method of proof is not quite so simple nor so elegant as the first. It uses the Weierstrass-Bolzano theorem, which itself is usually derived by a process of successive subdivisions.

(c) *Direct Proof by Subdivisions.* The two preceding proofs on account of their indirect character do not give a method for actually selecting the intervals required by the theorem. We can however use the subdivision idea to make such a selection. Let d_x be the least upper bound of the values of d for which the interval $(x-d, x+d)$ is interior to some interval of $\mathfrak{F}$. Then the function d_x is bounded from zero on the closed interval (a, b), i. e., there exists a d such that for all x of (a, b) $d_x > d > 0$. This can be deduced from the fact that d_x is a continuous function on a closed interval, or connected with the two results (a) that there exists for each x a vicinity of x such that d_x is bounded from zero in this vicinity, and (*b*) that for any function defined on (a, b) there exists a point x such that the greatest lower bound of f for each vicinity of x is the same as the greatest lower bound of f on (a, b). The final step in the proof then is to divide (a, b) into intervals of length less than d, each of which will be covered by some member of the family $\mathfrak{F}$ by the definition of d_x.

Observe that in this case we replace all the intervals to which a point is interior by a single member of another family, a family which is really involved in the proof of the uniform continuity theorem as given by Heine and by Dirichlet. The Pincherle result is also involved in the preceding proof, as well as the linear analog of Cousin's formulation of the Borel Theorem in the plane.* We might call attention to the following almost self-evident result involved in the preceding proof.

If every point of a bounded set E *is the middle point of an interval of length* $2d_x$*, and* d_x *is bounded from zero, then all the points of* E *are interior to a finite number of these intervals.*

(d) *Denumerable-to-Finite.*† When the given family is denumerable it is possible to select the finite family as follows. Suppose the intervals of $\mathfrak{F}$ are arranged in se-

* Cf. also Baire, ANNALI DI MATEMATICA, (3), vol. 3 (1899), pp. 13–15; Borel, COMPTES RENDUS, vol. 140 (1905), p. 299; Wirtinger, WIENER BERICHTE, vol. 108, IIa, pp. 1242–3.

† Cf. Young, W. H., PALERMO RENDICONTI, vol. 21 (1906), p. 127; Fréchet, PALERMO RENDICONTI, vol. 22 (1906), pp. 22–23.

quential order $I_1, I_2, \cdots, I_n, \cdots$. By a step-by-step process, we obtain an equivalent series, retaining those intervals which contain at least one point not interior to the preceding intervals of the array. Let $I_1, I_{n_2}, \cdots, I_{n_m}, \cdots$ be the members of the retained family. If this is finite, the theorem is proved. In the contrary case, we have an element x_m interior to I_{n_m} but not to the preceding intervals. By the use of the Weierstrass-Bolzano Theorem, this sequence contains a subsequence x_k' having a limit x_0 of (a, b). The contradiction arises from the fact that the interval I_{n_m} containing x_0 as an interior point will contain an infinite number of the x_k' as interior points, i. e., members of the sequence $\{x_m\}$ of index higher than n_m. We shall see later in the present paper that this method of proof is effective in general spaces.

(e) *Borel's Proof. The Lebesgue Chain.* The first proof of Borel also contains a scheme for selecting the finite subfamily, but it rests upon the properties of Cantor ordinal numbers, and the denumerability of the family, or of a family connected with the given family. Starting with the point a, there exists an interval I_1 containing a as interior point. If b_1 is the left hand end-point of I_1, let I_2 contain b_1 as interior point. Continuing thus, we get a sequence of intervals $I_1, I_2, \cdots, I_n, \cdots$. If the point b has not been reached, we get a limiting point b_∞ of the left hand end-points b_n and an interval I_∞ containing b_∞. By repeating the process, since the set of intervals is denumerable, we must eventually include the point b with an interval I_α where α is a transfinite ordinal of the second kind. We now extract from this well-ordered set of intervals a finite subset by noting that any interval I_α associated with a limit number α includes an infinite number of the end-points b_β of intervals preceding it. This enables us to select a decreasing array of ordinal numbers and such a decreasing set is finite.

This proof hinges on two facts, (a) that the system of intervals is denumerable, and (b) that a decreasing set of ordinal numbers selected from a well-ordered increasing set is finite.

While the denumerability of the family $\mathfrak{F}$ is postulated in this proof, the denumerability of the array $I_1, \cdots, I_\omega, \cdots$ leading to the point b can be deduced from the following fundamental theorem on intervals.

Any family of non-overlapping intervals is denumerable.

For the interval (0, 1), this follows from the fact that there exist at most n intervals of the set having a length greater than $1/n$, which gives a system for enumeration. Since the unbounded straight line can be broken up into a denumerable set of intervals of length unity, the extension of this result to any family of non-overlapping intervals is immediate.

In the preceding proof of the Borel Theorem the intervals $(b_\alpha, b_{\alpha+1})$ form a nonoverlapping family, which is therefore denumerable. Then the assumption that b would not be reached by a denumerable set of steps, would lead to a contradiction. In point of fact, there is contained implicitly in these considerations a theorem which has been called the Lebesgue Chain Theorem* which has been used by him in the discussion of lengths of curves. The theorem may be stated as follows.

LEBESGUE CHAIN THEOREM. *If the family $\mathfrak{F}$ of intervals I is such that to every point x of an interval (a, b) excepting perhaps b, there correspond intervals of the family having x as left-hand end-point, then there exists a finite or denumerable subfamily of these intervals without common points, containing each point of the interval, excepting possibly b, as an interior point or left-hand end-point.*

We shall give another proof of this theorem later on. Also we shall see that the idea of a Lebesgue chain and of applying a process of reduction via a decreasing set of ordinals underlies the proofs of other theorems similar to the Borel Theorem.

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* *Leçons sur l'Intégration*, p. 63.

(f) *Dedekind-Cut Proof.** Finally we call attention to a proof which depends on the Dedekind-Cut Axiom, i. e., is based on the properties of linear order. We observe that the point a is interior to an interval I of the family $\mathfrak{F}$. Let x' be the least upper bound of the points of (a, b) which can be reached with a finite number of intervals starting from a, i. e., x' is defined by a Dedekind cut. Now x' will belong to the interval (a, b) which is closed. Consequently there will be an interval $I_{x'}$ of the family $\mathfrak{F}$ to which x' is interior. It follows that x' is the point b.

An analysis of the proof shows that it rests upon the following principle.

INDUCTION PRINCIPLE FOR LINEAR ORDER. *Suppose a statement S satisfies the following conditions relative to an interval (a, b) (which may be the infinite interval $(-\infty, +\infty)$): (1) there exists a point of the interval for which S is true, (2) if the statement S is true for all points preceding x', then there exists a point y beyond x' for which S is true. Under these conditions S holds for the entire interval (a, b).*

A. Khintchine† has pointed out that this Induction Principle is logically equivalent to the Dedekind-Cut Axiom.

That the Dedekind-Cut Axiom implies the Induction Principle is practically contained in the above proof of the Borel Theorem. On the other hand, assume the truth of the Induction Principle. Suppose we have divided the points of the closed interval (a, b) into two groups A and B, each containing at least one point, and such that every point of A is less than (precedes) every point of B, and that A and B contain all the points of (a, b). Let the statement S be "The point x is a member of A." Since the conclusion of the Induction Principle is not holding, it

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* Lebesgue, *Leçons*, p. 105; O. Veblen, this BULLETIN, vol. 10 (1904), pp. 436–9 (see also TRANSACTIONS OF THIS SOCIETY, vol. 6 (1905), p. 167, where it is pointed out that the Borel Theorem applies to any well-ordered set); F. Riesz, COMPTES RENDUS, vol. 140 (1905), pp. 244–6.

† FUNDAMENTA MATHEMATICAE, vol. 4 (1922), pp. 164–6.

follows that either condition (1) or (2) is not holding. Now (1) is true, hence (2) must be false, i. e., there exists a point x' such that for every x preceding it the statement S is true, but S is not true for any point following x'. Since A and B contain all points of (a, b) it follows that the point x' belongs to one of these classes, i. e., is either the maximum of the class A, or the minimum of the class B.*

Along the same line is the observation that the Borel Theorem and Dedekind-Cut Axiom are logically equivalent; this remark is due to Veblen. The proof that the Borel Theorem has the Dedekind-Cut Axiom as a consequence, assumes that the interval is divided into the groups A and B, as conditioned in the preceding paragraph. With every point of the interval, we associate as far as possible an interval containing the point, and consisting only of points of group A or of group B. Either every point of (a, b) is interior to one of these intervals, or the contrary is true. In the first case, we have by the Borel Theorem a finite number of intervals, reaching from a to b, containing every point as an interior point. Since a belongs to A, the intervals overlap, and each interval contains only points of A or B, it follows that every point of (a, b) is a point of A, which is contrary to hypothesis. Hence there exists at least one point which is not interior to an interval containing only points of A, or of B, i. e., satisfies the conditions of the Cut Axiom. An analysis of this proof shows that the theorem is a close relative of the following theorem.

If a function is not invariant in sign throughout an interval (a, b), *then there exists a point of the interval in every vicinity of which the function is not invariant in sign.*†

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* To be a complete analog to mathematical induction, condition (2) should read: If S is true for all points preceding x', then it is true for x'. In this form, however, it does not seem to have the power which carries one to the end of the intervals.

† K. P. Williams, ANNALS OF MATHEMATICS, (2), vol. 17 (1915–6), pp. 72–3.

The Induction Principle also furnishes a method of proof for the Lebesgue Chain Theorem. Two have been given along similar lines. The first, due to J. Pal,* considers the case in which to every point of (a, b) there corresponds only a single interval having this point as left-hand end-point. In that case, any chain beginning from a is necessarily unique. The second, due to G. C. Young,† applies to the general case, where the number of intervals associated with a given point is not specified. Obviously in this case there may be many different chains leading from one point to another.

We say that the set $\mathfrak{S}$ of intervals I forms a chain from the point a to the point x in case $\mathfrak{S}$ is a set of non-overlapping intervals such that every point of (a, x), excepting perhaps x, is either an interior point or a left-hand end-point of one of the intervals of $\mathfrak{S}$. The point x may be interior to an interval, or an end-point of a chain, i. e., a right-hand end-point of an interval, or the limiting point of a sequence of intervals of the chain $\mathfrak{S}$.

In applying the Induction Principle assume that the point x is such that for every y less than x, there exists a chain from a to y. The case in which there is an end-point y of a chain from a to y, such that there exists an interval with y as left-hand end-point which contains x, and the case in which x is an end-point of a chain leading from a to x, are easily disposed of. If neither of these two cases are holding, let $e_1, e_2, e_3, \cdots, e_n, \cdots$ be a monotonic decreasing sequence of positive numbers converging to zero. Then by hypothesis there exists a chain from a to x_1, where the distance of x_1 to x is less than e_1. If there exists a chain extending from x_1 beyond x, then by adding this chain to the chain from a to x_1, we extend beyond x. In the contrary case, let y_1 be the least upper bound of points reached by chains beginning at x_1. Then there exists a chain from x_1

* PALERMO RENDICONTI, vol. 33 (1912), pp. 352–3.

† BULLETIN DES SCIENCES MATHÉMATIQUES, (2), vol. 43 (1919), pp. 245–7.

to a point x_2 with $y_1 - x_2 < e_2$. By a repetition of this process, we get a sequence of points $x_1, x_2, x_3, \cdots$ having a limiting point x_0, and by the method of construction, by combining chains, we get a chain from a to x_0. Now x_0 is a left-hand end-point of an interval of the family $\mathfrak{F}$ whose length is greater than e_n for n sufficiently large. But this interval added to the chain from a to x_0 would give us a chain reaching beyond y_n for n sufficiently large, contrary to the definition of y_n. It follows that there is a chain from x_1 to x, and so beyond x.

A careful analysis of the proof shows that its basis is really the same as the proof of this theorem using the Cantor numbers, viz., the fact that there are at most a denumerable set of non-overlapping intervals.

3. *Extensions of the Borel Theorem.* (a) *To Closed Sets.* An almost immediately obvious extension of the Borel Theorem is to replace the closed interval covered by the family $\mathfrak{F}$ by a closed set of points E. The methods of proof sketched above all apply excepting that in the case of (e) and (f) which involve order on a line, it may be necessary to take account of the intervals complementary to the closed set. An alternative method of procedure is to take the smallest interval (a, b) containing the set E, and enlarge the family $\mathfrak{F}$ by the addition of the intervals belonging to the complement of E with respect to (a, b) and then apply the theorem for the interval.*

(b) *Extension to n-Dimensional Space.*† A further extension which is possible is to n-dimensional space. This was conceived almost as soon as the Borel Theorem, due to the consideration of functions of the complex variable.‡

* Cf. also W. H. Young, PROCEEDINGS OF THE LONDON SOCIETY, vol. 35 (1902), pp. 387–8.

† Cf. Schoenfliess-Hahn, *Entwickelungen*, vol. 1, 1913, pp. 239–241.

‡ Cf. Cousin above. The Cauchy-Goursat theorem is virtually based on a two-dimensional Borel Theorem. See TRANSACTIONS OF THIS SOCIETY, vol. 1 (1900), pp. 14–16.

We define the point x *interior to a set of points I of n-dimensional space* to mean that there exists a vicinity of x (i. e., an n-dimensional sphere or cube having x as center) containing only points of I. Then the Borel Theorem reads as follows.

If E is any closed set of n-dimensional space covered by a family $\mathfrak{F}$ of sets I (i. e., such that every point of E is interior to some member of the family) then a finite subfamily of $\mathfrak{F}$ covers E.

Some of the proofs given for the linear interval are immediately extensible to space. This is true of the methods (a) and (b) by successive subdivisions, and (c) in which the family $\mathfrak{F}$ is replaced by a family of spheres with the points of E as centers, also of (d) the denumerable case. (e) and (f) seem to use particularly linear order and consequently are effective mainly in proofs by induction, passing from the case of n-dimensional to $(n+1)$-dimensional space. Lebesgue* suggests an ingenious method of passing from the plane to the linear interval by using the Peano curve which maps the square on the linear interval.

4. *Necessary Conditions. Lindelöf Theorem.* The hypothesis of the Borel Theorem specifies as sufficient conditions that the set E be closed and bounded. These conditions are also necessary. If E is not closed then there exists a point x_0 limiting point of E not belonging to E. If we surround each point x of E by a sphere of radius one-half of the distance of x from x_0, then no finite subfamily of this family of spheres will cover E. The fact that E must be bounded is obvious.

In the same direction is the remark that the Weierstrass-Bolzano Theorem is a consequence of the Borel Theorem. For suppose an infinite set E_0 of points $x_1, \cdots, x_n, \cdots$ contained in a bounded closed region E. Then either for every point x of E there exists a sphere around x containing at most one point of E_0, or there exists a point x such that

* *Leçons*, p. 119.

every sphere about x contains an infinity of points selected from E_0. In the first case, the spheres constitute a family $\mathfrak{F}$ for E to which the Borel Theorem applies. But a finite number of spheres of $\mathfrak{F}$ contain only a finite number of points of E_0. Hence E_0 has a limiting point in E.

We note in passing that it is immaterial whether the family $\mathfrak{F}$ consist of open sets (containing only interior points) or arbitrary sets of points to make the Borel Theorem valid.

Returning to the linear case, the fact that the Borel Theorem does not hold for sets in general suggests the question what can be said about the set of points interior to any family of intervals. If by the term *two families of intervals are equivalent* we mean that they cover the same set of points, then we have on the one hand

Any family of intervals is equivalent to a family of non-overlapping intervals.

This is immediately evident. We need only take any point interior to some interval and proceed to the left and right until we meet a point which is not interior to any interval of the given family. In this way we define a group of non-overlapping sets which is denumerable.

As an analog to the Borel Theorem we have the following theorem, which is due to Lindelöf.*

LINDELÖF THEOREM. *In any family of intervals it is possible to find a denumerable sub-family having the same interior points.*

This theorem has been proved in different ways. Mention might be made of the following proofs:

(a) *Using Density of Rational Points on a Line.*† Let x be any point interior to some interval I of the family. Then there exists an interval R with rational end-points containing x, and entirely interior to I. This sets up a correspondence between a family of the given intervals and the

* COMPTES RENDUS, vol. 137 (1903), p. 697.

† Cf. W. H. Young, PALERMO RENDICONTI, vol. 21 (1906,) p. 125.

family of intervals with rational end-points, which is denumerable.

We observe that this method of proof makes use of the density of the rational points on a line and their denumerability. An equivalent method of constructing the intervals with rational end-points, would be to use the rational points interior to the family of intervals, and make each of them the mid-point of an interval with rational end-points interior to some interval of the set. In this form, the proof of the corresponding theorem in n-space can be made.

(b) *Via the Borel Theorem.* Let (a, b) be one of the intervals of the equivalent family of non-overlapping intervals, and x_0 any point of (a, b). Let $x_0, x_{-1}, x_{-2}, \cdots$ be a monotonic sequence of points approaching a, and $x_0, x_1, x_2, \cdots$ be a monotonic sequence approaching b. Then the closed intervals (x_{-n}, x_n) can be covered by a finite number of intervals of the given family $\mathfrak{F}$. This gives us a method for selecting a denumerable set of intervals having the desired property relative to (a, b). The final result is a consequence of the properties of denumerability.

(c) *Lindelöf Proof.* Suppose the points covered by $\mathfrak{F}$ belong to a finite interval (a, b). For any x covered by $\mathfrak{F}$, let d_x be the maximum of the values of d for which $(x-d, x+d)$ is interior to one of the intervals of the family $\mathfrak{F}$, and consider the set E_n of points of the set E covered by $\mathfrak{F}$, for which $d_x > 1/n$. Then by the observation of §2(c), it follows that the set E_n can be covered by a finite number of intervals chosen from the family $\mathfrak{F}$. This gives a method for the enumeration of the subfamily. The extension to the unbounded interval results from the fact that it can be divided into a denumerable set of finite intervals.*

A similar method of procedure applies in n-space.

* W. H. Young, Palermo Rendiconti, vol. 21 (1906), pp. 126–7, gives another proof utilizing somewhat similar procedure, which resulted in a series of polemics with Schoenflies, the final shot being fired by the latter in Palermo Rendiconti, vol. 35 (1913), pp. 74–78. At best the method of proof, even as validated by Schoenflies, is not satisfactory.

5. *Strict Families of Intervals.** It is obvious that in general the selection of the finite subfamily of intervals in the Borel Theorem may be made in many different ways. Obviously too, some of the intervals of the finite family may be unnecessary in that all points interior to them are interior to other intervals of the family. We shall call a *strict* family of intervals, a family in which each interval is necessary, in the sense that it contains a point not interior to any of the other members of the family. We have then

Any finite family of intervals can be replaced by an equivalent strict family.

We note first that if three intervals have a common point they can be replaced by at most two of these intervals. Let the intervals be (a_1, b_1), (a_2, b_2), and (a_3, b_3), where since they contain a common interior point, the notation is chosen so that

$$a_1 \leqq a_2 \leqq a_3 \leqq b_1 .$$

Then obviously if $b_2 < b_3$, we can dispense with (a_2, b_2) and if $b_2 > b_3$, then we can dispense with (a_3, b_3). By the use of this result the theorem stated is immediate. The process of deletion in any particular case may be tedious, especially if governed by other considerations, such as, for instance, the desire to make the sum of the lengths of the retained intervals a minimum.

More generally we have the following theorem.

If any family of intervals is such that every point interior to one of the intervals is interior to at most a finite number of intervals of the family, then we can select a strict subfamily, equivalent to the given family.

We note in the first place that the given family is necessarily denumerable. For to each point interior to an interval of the family there will correspond a vicinity which is common to a definite finite number n_x of intervals. By

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* Cf. Denjoy, Journal de Mathématiques, (7), vol. 1 (1915), pp. 223–30.

the Lindelöf Theorem these vicinities can be replaced by a denumerable set of the same vicinities, and since to each of the final vicinities there corresponds a finite number n_x it follows that the original family of intervals is denumerable.

Let then the intervals be arranged in sequential array $I_1, \cdots, I_n, \cdots$. Let I_{n_1} be the first interval of the array covering a point not interior to any of the succeeding intervals. Such an interval will always exist. We determine I_{n_m} as the first interval of the sequence following $I_{n_{m-1}}$ containing a point not interior to $I_{n_1}, \cdots, I_{n_{m-1}}$, and a point not interior to any of the intervals following I_{n_m}. It is obvious that the resulting family is a strict family. It remains to show that every point x covered by the original family is covered by the subfamily. Since x is an interior point of a finite number of intervals of $\mathfrak{F}$ the indices of the sequence $I_1, \cdots, I_n$ to which x is interior have a definite maximum N, i. e. there will be an index $n_m < N$ such that x is interior to I_{n_m}.

From the point of view of measure, a strict family of intervals has the property that the sum of the lengths of the intervals is less than double the sum of the lengths of the intervals covered. For consider any strict family of intervals. We note first of all that it is necessarily denumerable. For since each interval contains a point not interior to other intervals it contains a subinterval having no points in common with other intervals. These subintervals define a system of non-overlapping intervals which is denumerable, and consequently the original family, which is in one-to-one correspondence with it, is denumerable. Let x_n be a point interior only to I_n. Then the points x_n have as limiting points only points not interior to any interval of the family. From the contrary assumption would follow that some of the points x_n are interior to more than one interval of the given family. As a consequence the points x_n in each interval (a_k, b_k) of the equivalent family of non-overlapping intervals can be arranged in order so that between two points no further points of the sequence appear. Consequently no point of (a_k, b_k) appears in more than two intervals of the family.

Denjoy has given a condition under which the reduction of a family to an equivalent strict subfamily is possible. If a family $\mathfrak{F}$ of intervals is *upper semi-closed* in case every interval which is the limit of a sequence of intervals I_n chosen from the family is part of an interval of or belongs to $\mathfrak{F}$, then we have the following theorem.

From every upper semi-closed family $\mathfrak{F}$ of intervals it is possible to select an equivalent strict subfamily.

It will obviously be sufficient to show the possibility of selecting an equivalent subfamily $\mathfrak{F}_0$, such that every point covered by $\mathfrak{F}$ is interior to at most a finite number of intervals of $\mathfrak{F}_0$. Let (a, b) be an interval of the equivalent family $\mathfrak{G}$ of non-overlapping intervals. There are three possibilities:

(a) The points a and b are both end-points of intervals of $\mathfrak{F}$. Then, applying the Borel Theorem, we get a finite sub-family having every point of (a, b) excepting a and b as interior points.

(b) Both a and b are not end-points of intervals of $\mathfrak{F}$. Then by successive applications of the Borel Theorem, as in the proof of the Lindelöf Theorem (§3(b)), we construct an equivalent family which has the property that a or b is a limiting point of end-points of any infinite set of intervals selected from this family. If possible let x be an interior point of (a, b) interior to an infinite number of intervals of the resulting subfamily. Then a or b is a limiting point of end-points of these intervals. By the semiclosure of the family $\mathfrak{F}$ it follows that a or b is then an end-point of an interval reaching at least to x, contrary to the assumption. Hence every ppint interior to (a, b) is interior to at most a finite number of intervals of this subfamily.

(c) If only one of the end-points a or b is an end-point of an interval of $\mathfrak{F}$, then a process similar to case (b) utilizing only one end-point will apply.

As a corollary of this result we have the following theorem.*

If a family $\mathfrak{F}$ of intervals is such that for every e there are at most a finite number of intervals of $\mathfrak{F}$ of length greater than e, then there exists an equivalent strict subfamily of $\mathfrak{F}$.

6. *Other Theorems on Reduction of Families of Intervals to Finite Subfamilies.* We turn our attention briefly to a number of analogs of the Borel Theorem, most of which are due to W. H. and G. C. Young.† A theorem which has the Borel Theorem as a consequence, but for which the converse has not yet been shown, has been called by the Youngs the Heine-Young theorem, because of its similarity to Heine's proof of the uniform continuity theorem.

HEINE-YOUNG THEOREM. *With every point x of a closed interval (a, b) there are associated two intervals, an R_x having x as left-hand end-point, and L_x having x as right-hand end-point. These intervals are connected by the condition that if x' is interior to the L_x for x, then $R_{x'}$ contains x as an interior point or an end-point. Then a finite number of the R intervals cover (a, b) without overlapping.*

The R intervals without the intervention of the L intervals are equivalent to a Lebesgue chain, and since the R_x is unique for every point there will be only one such chain. The presence of the L intervals insures the finiteness of the chain by preventing limiting points. For the L corresponding to a possible limiting point x would include an infinite number of x' whose $R_{x'}$ by hypothesis should reach up to or beyond x.

The finiteness of the R chain is also apparent from another point of view. The L intervals are equivalent to a chain. Now any L is covered by at most two R intervals. We can then as in the case of the Borel Theorem define a sequence

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* Cf. R. L. Moore, PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 10 (1924), 466–7.

† Cf. *Reduction of intervals*, PROCEEDINGS OF THE LONDON SOCIETY, (2), vol. 14 (1915), pp. 111–130.

of decreasing ordinals by beginning with R_a. For if a point x' is a limiting point of the L intervals, of the chain, the corresponding $R_{x'}$ will cover an infinite number of the L intervals, and have as right-hand end-point a point interior to an L or an end-point of an L or a point which is again a limiting point. In either of these cases we can proceed in the formation of our decreasing ordinal series, which is finite.

While this theorem seems to be more general than the Borel Theorem, its usefulness is rather hampered by the peculiar way in which the two sets of intervals are interlaced.

Another theorem of the same type is due to Lusin*, which is stated by him as applying to non-dense perfect sets.

LUSIN THEOREM. *Let E be any non-dense perfect set, and $\mathfrak{G}$ the family of open intervals (a_k, b_k) complementary to E relative to an enclosing interval (a, b). Moreover, suppose that to every point x of E not an a_k or b_k there correspond intervals of a family $\mathfrak{F}$ to the right of x of the form (x, a_k), and all intervals to the left of the form (b_k, x) in a certain neighborhood of x; while to the points b_k, only intervals of first type correspond, and to points a_k only intervals of the second type; then a finite number of the intervals from $\mathfrak{F}$ and $\mathfrak{G}$ cover (a, b) or E without overlapping.*

The proof can be made along the lines of the Young Theorem; the right-hand intervals together with the intervals of $\mathfrak{G}$ can be formed into a chain, this being reduced to a finite set by the inclusion of the left-hand intervals. From this it is apparent that to prevent overlapping it is not sufficient to assume as Lusin does that only a single interval the right and left corresponds to each point of E.

W. H. and G. C. Young credit the following theorem for a closed interval to Lusin.

If to every point x of a closed interval (or closed point set) there correspond all the intervals to the left and right of the

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* MOSCOW MATHEMATICAL SOCIETY TRANSACTIONS (SBORNIK), vol. 28 (1911–12), p. 270.

*point in a given neighborhood of the point, then a finite number of these intervals suffice to cover the interval (or point set) without overlapping.**

With every x of (a, b) we associate an interval of a new family such that x is the middle point of its interval, and the interval consists of an L and an R of equal length selected from intervals associated with the point. A finite number of these intervals suffice to cover (a, b). We assume that the intervals retained give a strict covering for (a, b). Then if I_{n_1} contains a, let x_1 be its middle point. If $x_1 > a$, we reach x_1 by noting that (a, x_1) belongs to the given family. Let I_{n_2} overlap with I_{n_1} and x_2 be its middle point. Then $x_2 > x_1$. If x_2 belongs to I_{n_1} then x_1 and x_2 is an interval of the given family. If x_2 is not in I_{n_1} then we reach x_2 by taking the intervals $x_1\ b_1$ and $b_1\ x_2$, where b_1 is the right-hand end-point of I_n. It is obvious that in this way we can construct a finite number of intervals as required.

A slightly more general theorem can be obtained by associating with every point all the intervals to the left in a certain vicinity, and only one or more to the right. In this form the proof given above via the Borel Theorem is not valid, and it is not clear whether the Borel Theorem can be used. It can be deduced by using the Lebesgue chain associated with the intervals to the right; or, following the Youngs, we get an equivalent method by associating with every point x of (a, b) as an R_x the smallest interval containing all the intervals of the given family having x as left-hand end-point and as L_x the given vicinity to the left. Then the Heine-Young Theorem applies; there exists a finite number of the R intervals reaching from a to b. Any of these R intervals can be replaced by at most two intervals of the original family.

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* This theorem can be made the basis of a proof of the theorem *If $f(x)$ has a Riemann integrable derivative $f'(x)$ on (a,b) then* $f(b)-f(a)=\int_a^b f(x)dx$, without using the mean-value theorem of the differential calculus, a desideratum in considerations in general functional space.

In trying to avoid the infinite elements which pervade the Lebesgue Chain Theorem, the Youngs state the following lemma.

YOUNG'S LEMMA. *If to every point of the closed interval (a, b) there corresponds an interval of a family $\mathfrak{F}$ having this point as left-hand end-point, then for every e, there exists a finite subfamily $\mathfrak{F}_e$ of non-overlapping intervals, such that he sum of the complementary intervals is less than e.*

If we assume the Lebesgue Chain Theorem this result is immediate. The Youngs prove the theorem by an ingenious application of the Heine-Young Theorem. Chose n so that

$$2(b-a) < ne.$$

Then to every point x of (a, b) we make correspond as L_x an interval to the left of x whose length is $1/n$ of the least upper bound of the intervals of $\mathfrak{F}$ associated with x. If $(x, x+h_x)$ is any interval of $\mathfrak{F}$, then we add to $\mathfrak{F}$ all the intervals $(y, x+h_x)$ where

$$x - \frac{h_x}{n} \leqq y \leqq x.$$

Let R_x be the smallest interval containing all the intervals of this extended family which have x as left-hand end-point. Then it is apparent that the R and L intervals satisfy the conditions of the Heine-Young Theorem, so that a finite number of the R intervals extend from a to b without overlapping. The result desired follows from the fact that each R can be approximated up to $2/n$ of its length by an interval of the family $\mathfrak{F}$.

It is obvious that the Young Lemma will hold also for closed sets of points E, in which form it was originally stated.* A proof similar to this other proof given by Young has been made by Sierpinski† for the following more general theorem which utilizes the properties of upper measure.

* PROCEEDINGS OF THE LONDON SOCIETY, (2), vol. 9 (1911), pp. 325ff.

† Cf. FUNDAMENTA MATHEMATICAE, vol. 4 (1923), pp. 201–3.

If E is any bounded linear set, and $\mathfrak{F}$ a family of intervals such that to every x of E there corresponds an interval R_x of $\mathfrak{F}$, having x as left-hand end-point, then for every e it is possible to determine a finite subfamily $\mathfrak{F}_e$ of $\mathfrak{F}$, consisting of non-overlapping intervals, and such that

$$\overline{m}(E - E \cdot F_e) < e.*$$

The proof is as follows. Let E_n be the set of points of E for which there exists an R_x of length greater than $1/n$. Then the E_n form a monotonic increasing family of sets such that

$$E = \sum E_n,$$

and consequently

$$\lim_n \overline{m}E_n = \overline{m}E.$$

Chose n so that

$$\overline{m}E - \overline{m}E_n < \frac{e}{2}.$$

Let (a_1, b_1) be the smallest interval containing E_n, its length being l. Obviously if we use intervals of $\mathfrak{F}$ of length greater than $1/n$, then there will be at most nl possible non-overlapping intervals in (a_1, b_1). If between every two intervals we allow a space d such that

$$nld < \frac{e}{2},$$

then the points of E_n not covered will be of upper measure at most $e/2$. Since a_1 is a lower bound of points of E_n there will be a point x_1 of E_n in the interval (a_1, a_1+d) and an R_x of length greater than $1/n$. Let a_2 be the lower bound of the points of E_n to the right of $R_{x_1} = R_1$. Take x_2 so that x_2 belongs to (a_2, a_2+d) and E_n. Continuing in this manner we get a finite number of intervals $R_1, \cdots, R_m$. Let S be the set of points belonging to $R_1, \cdots, R_m$. Then we wish to show that

$$(\overline{m}E - ES) < e.$$

* We follow the usual notations: $\overline{m}E$ or $\overline{\text{meas}}\, E$ for upper measure of E; $E \cdot F$ the set of points common to E and F; $E+F$ set of points belonging to either E or F; $E-F$ the sets of points of E not in F.

Now by the properties of upper measure we have*

$$\bar{m}E = \bar{m}(E - ES) + \bar{m}ES.$$

By the selection of E_n, we have

$$\bar{m}E - \bar{m}E_n < \frac{e}{2},$$

and by the selection of S,

$$\bar{m}E_n - \bar{m}E_nS < \frac{e}{2},$$

so that

$$\bar{m}E - \bar{m}E_nS < e.$$

But E contains E_n and so ES contains E_nS. Hence

$$\bar{m}(E - ES) = \bar{m}E - \bar{m}ES < \bar{m}E - \bar{m}E_nS < e.$$

These theorems are of value in connection with the discussion of the distribution of infinite derivatives of a function of a single variable. The Young Lemma and the Sierpinski extension have been used by the Youngs† and by Rajchman and Saks‡ to obtain in a simple way forms of the theorem that any monotonic function and therefore any function of bounded variation has a finite derivative excepting at a set of points of measure zero.

Closely related to these theorems is a so-called "tile" theorem of the Youngs. If to a point correspond all the intervals in a certain vicinity of the point, then any such interval is called a *tile* and the point is called *the point of attachment.*

TILE THEOREM.§ *Suppose a family of intervals $\mathfrak{F}$ such that to every point x of a linear bounded set E, there correspond all the intervals in a certain neighborhood of x. Then for*

* Cf. Hausdorff, *Mengenlehre*, p. 415.

† PROCEEDINGS OF THE LONDON SOCIETY, (2), vol. 9 (1911), pp. 325–35.

‡ FUNDAMENTA MATHEMATICAE, vol. 4 (1923), pp. 204–13.

§ PROCEEDINGS OF THE LONDON SOCIETY, (2), vol. 2 (1904), pp. 67–9; ibid., (2) vol. 14 (1915), pp. 122–126. The Youngs assume that E is measur able.

every e and d, there exists a finite or denumerably infinite subfamily $\mathfrak{F}_{de}$ of $\mathfrak{F}$, such that

(*a*) *every interval I of $\mathfrak{F}_{de}$ is of length less than d,*

(*b*) *the point associated with each interval of $\mathfrak{F}_{de}$ is interior only to that interval,*

(*c*) *every point of E is interior to some interval of $\mathfrak{F}_{de}$, and*

(*d*) $$\sum mI_n - \overline{m}E < e$$

where I_n are the intervals of $\mathfrak{F}_{de}$.

In case the set E is a closed interval, the set $\mathfrak{F}_{de}$ is finite.* In this case the proof can be made via the Borel Theorem. We discard all intervals of $\mathfrak{F}$ of length greater than d, and then associate with each point x an interval of the remaining family associated with x, of which x is the middle point. Then replace this new family by a finite strict subfamily via the Borel Theorem. Let $x_1, \cdots, x_n$ be the mid-points of the resulting intervals arranged in order. We can then select intervals from $\mathfrak{F}$ attached to the points $x_1, \cdots, x_n$ in such a way that the overlap lies entirely between x_i, x_{i+1} and has for each i a length less than e/n.

In the case of any set E, we enclose E in a set of non-overlapping intervals J_n such that

$$\sum mJ_n - \overline{m}E < e.$$

Discard all the intervals of $\mathfrak{F}$ of length greater than d, and those which do not lie completely interior to some J_n. Associate with each point x of E an interval of the remaining intervals associated with x, having x as middle point, forming a family $\mathfrak{F}_0$. The family $\mathfrak{F}_0$ is equivalent to a denumerable family of non-overlapping intervals K_m lying interior to the J_n, which contains E, so that

$$\sum mK_m - \overline{m}E < e.$$

* Also true if E is a closed set. Obviously this case is closely related to the Lusin-Young Theorem.

Now by following a process similar to that used in §5, and by discarding, at each stage, intervals not needed, we can reduce $\mathfrak{F}_0$ to an equivalent strict family $\mathfrak{F}$ of intervals. This strict family of intervals can be replaced by a family $\mathfrak{F}_{de}$ of intervals chosen from $\mathfrak{F}$, satisfying the conditions (*b*) and (*d*), by a line of reasoning similar to that used in connection with the case where E is a closed interval.

7. *The Vitali Theorem.* The tile theorem of the Youngs is very closely related to the Vitali Theorem, which plays a role in connection with measurable sets and derivatives of functions comparable to that of the Borel Theorem relative to closed sets and continuous functions. It was stated by Vitali* for the linear interval in a form equivalent to the following:

If $\mathfrak{F}$ is a family of intervals such that for every point x of the bounded measurable set E, there exists a set of intervals of $\mathfrak{F}$ containing x, whose lengths approach zero, then there exists a denumerable subfamily of $\mathfrak{F}$ consisting of non-overlapping intervals, the sum of whose lengths is greater than the measure of E.

This theorem has been extended in various ways, especially to higher dimensions, the chief extensions being due to Lebesgue† and Carathéodory.‡ Probably the simplest statement and proof of the fundamental extended form of the theorem has been given by Banach,§ viz.

VITALI THEOREM. *Let E be any bounded set of points in a space of n dimensions. Let $\mathfrak{F}$ be a family of closed sets of points I, such that to each point x of E, there corresponds a sequence I_n chosen from $\mathfrak{F}$ subject to two conditions:*

(1) *if $r_n(x)$ is the radius of the smallest sphere $C_n(x)$ of center x containing $I_n(x)$, then $\lim_n r_n(x)=0$, and*

* ATTI DI TORINO, vol. 43 (1907), pp. 229–236.

† ANNALES DE L'ÉCOLE NORMALE, (3), vol. 27 (1910), pp. 391–5.

‡ *Vorlesungen über reelle Funktionen*, 1918, pp. 299–307.

§ *Sur le théorème de Vitali*, FUNDAMENTA MATHEMATICAE, vol. 5 (1924), pp. 130–6.

(2) *there exists a quantity* α *independent of* n *and* x *such that*

$$\frac{\text{meas}\, I_n(x)}{\text{meas}\, C_n(x)} > \alpha .$$

Then for every e, *there exists a finite or denumerable family of sets* $I_n(x_n) = I_n'$ (1) *without common points, and such that*

$$\sum \text{meas}\, I_n' < \overline{\text{meas}}\, E + e \tag{2}$$

and

$$\overline{\text{meas}}\, (E - \sum I_n' E) < e . \tag{3}$$

In this theorem it is immaterial whether the sets $I_n(x)$ contain x.* Also it is possible to replace the spheres $C_n(x)$ by cubes having x as center, or rectangular parallelepipeds, the ratios of whose dimensions are bounded from infinity and zero. That the theorem is not true with an unconditioned set of rectangular parallelepipeds has been shown by Banach.†

We give the proof in two dimensions, the changes to be made for the n-dimensional case being obvious.

Since E is bounded, it is possible to find an open set U containing all the points of E and such that

$$\text{meas}\, U \leqq \overline{\text{meas}}\, E + e .$$

We then reduce the family $\mathfrak{F}$ to the family $\mathfrak{F}_1$, by retaining only those $I_n(x)$ which are contained in U, so that condition (2) of the conclusion is fulfilled if the selection can be made in accordance with condition (1).

Let k be any constant greater than unity. Then by selecting an

$$I'_1 = I_{n_1}(x_1) ,$$

such that the radius $r_{n_1}(x_1)$ is sufficiently near to the least upper bound of the radii $r_n(x)$ we can make sure that

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* In Lebesgue's formulation (loc. cit., p. 391) the $I_n(x)$ contain x but then the spheres containing $I_n(x)$ need not have x as center.

† Loc. cit., pp. 134–6.

$$r_n(x) < kr_{n_1}(x_1) = kr(I_1'),$$

for every n and x. Similarly if $I_1', \cdots, I_{m-1}'$ have been determined, then by a similar method we can select the set

$$I_m' = I_{n_m}(x_m)$$

with properties (a) I_m' does not have any points in common with I_j' for $j \leqq m-1$, and (b)

$$r_n(x) < kr_{n_m}(x_m) = kr(I_m')$$

for all $I_n(x)$ which have no points in common with I_j' for $j \leqq m-1$. Since E and therefore also U is bounded, and consequently has finite upper measure, it follows that meas I_m' approaches zero with m and consequently $r(I_m')$ approaches zero with m, due to condition (2) of the hypothesis. Let C_m' be the circle of radius $(2k+1)\,r(I_m')$, center x_m. Then we show that for every m, all points of E belong to

$$\sum_1^{m-1} I_i' + \sum_m^{\infty} C_j'.$$

We observe first that the sets I_n' for $n \geqq m$ are chosen from the reduced family $\mathfrak{F}_m$ of $\mathfrak{F}$ with respect to the open set

$$U_m = U - \sum_1^{m-1} I_j'$$

as I_n', for $n \geqq 1$, was selected from the reduced family $\mathfrak{F}_1$ with respect to U. Let x_0 be any point of E. Then either x_0 belongs to some I_j' or it belongs to U_m for every m. Let $I_m(x_0)$ be the I of maximum radius contained in U_m. Then since $r(I_n)$ approaches zero with n, there exists a value of n such that

$$r(I_m(x_0)) > kr(I_n'),$$

but

$$r(I_m(x_0)) \leqq kr(I_j'),$$

for $j = m, \cdots, n-1$. It follows that $I_m(x)$ must have parts in common with one of the sets $I_m', \cdots, I_{n-1}'$, and hence is completely covered by the corresponding C_i' by the method of formation of the C_i'.

Now since

$$\sum_1^\infty \text{meas}\, I_i' < \overline{\text{meas}}\, E$$

it follows that $\sum_m^\infty$ meas I_i' approaches zero with m. Hence since

$$\sum_m^\infty \text{meas}\, I_i' > \alpha \sum_m^\infty \text{meas}\, C_i' = \frac{\alpha}{(2k+1)^2} \sum_m^\infty \text{meas}\, C_i'$$

it follows that $\sum_m^\infty$ meas C_i' approaches zero with m, i.e., the set $\sum_1^\infty I_i'$ contains all points of E up to a set of measure zero.

The following generalization is possible.*

The Vitali Theorem is still valid in case the set E is any set, and the constant α of condition (2) *of the hypothesis is dependent on x, but independent of n.*

The case when E is any set can be reduced to the case when E is bounded, by consideration of the fact that n-dimensional space can be broken up into a denumerable set of compartments of finite magnitude. For the case where E is bounded and α is dependent on the point x, we consider the sets E_n of points of E which satisfy the conditions

$$\frac{1}{n+1} < \alpha(x) \leqq \frac{1}{n}$$

where $\alpha(x)$ is the greatest lower bound of the values of the fraction

$$\frac{\text{meas}\ I_m(x)}{\text{meas}\ C_m(x)}.$$

We can then apply the previous theorem to the sets E_n, successively. We determine $I_1', \cdots, I_n'$ so that

* Lebesgue, loc. cit., p. 393; Carathéodory, loc. cit., p. 305.

$$\text{meas}\left(E_1 - \sum_{j=1}^{n} E_1 I_j'\right) < \frac{e}{2},$$

then $I'_{n_1+1}, \cdots, I_{n_2}$ belonging to $U - \sum_{j=1}^{n_1} I_j'$, so that

$$\text{meas}\left(E_1 + E_2 - \sum_{j=1}^{n_2} (E_1 + E_2) I_j'\right) < e\left(1 - \frac{1}{2^2}\right),$$

and so on.

The Vitali Theorem gives a very elegant method for demonstrating the following generalization of the Lebesgue metric density theorem. For the statement of the theorem we define the upper metric density of a set E at a point x of E as

$$\lim_{r\to 0} \frac{\overline{\text{meas}}\, C_r E}{\text{meas}\, C_r}$$

where C_r is a sphere of radius r and center x. Then the theorem is*

The points of any set E at which the metric density is not unity form a set of zero measure.

For let E_0 be the set of points of E at which

$$\lim_{r\to 0} \frac{\overline{\text{meas}}\, C_r E}{\text{meas}\, C_r}$$

does not exist or is less than unity, and let E_k, k a positive integer, be the points of E for which there exists a sequence of circles with radii r_n converging to zero, such that

$$\overline{\text{meas}}\, C_{r_n} E < \left(1 - \frac{1}{k}\right) \text{meas}\, C_{r_n}.$$

Then $E_0 = \sum E_k$. The spheres C_{r_n} form a family $\mathfrak{F}$ as required by the Vitali theorem for the points of E_k. Hence for every e there exists a denumerable set of the C_{r_n}, C_m' satisfying the conditions

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* See Lebesgue, loc. cit., p. 407; H. Blumberg, TRANSACTIONS OF THIS SOCIETY, vol. 24 (1923), pp. 122ff; and W. Sierpinski, FUNDAMENTA MATHEMATICAE, vol. 4 (1923), pp. 167–171, where other references are to be found.

$$\overline{\text{meas}}\, E_k = \overline{\text{meas}} \sum_m C_m' E_k = \sum_m \overline{\text{meas}}\, C_m' E_k,$$

and

$$\sum_m \text{meas}\, C_m' < \overline{\text{meas}}\, E_k + e.$$

But since E_k is part of E, we have

$$\overline{\text{meas}}\, C_m' E_k < \overline{\text{meas}}\, C_m' E < \left(1 - \frac{1}{k}\right) \text{meas}\, C_m'.$$

Then

$$\overline{\text{meas}}\, E_k = \sum_m \overline{\text{meas}}\, C_m' E_k < \left(1 - \frac{1}{k}\right) \sum_m \text{meas}\, C_m'$$
$$< \left(1 - \frac{1}{k}\right)\left(\overline{\text{meas}}\, E_k + e\right),$$

from which

$$\overline{\text{meas}}\, E_k < (k - 1)e,$$

i.e. E_k is of measure zero for every k, giving the same result for E_0.

The question naturally arises what it is possible to do in the matter of selecting the sets I_n so as to contain all points of E. H. Rademacher* has given the following which might be considered a generalization of the Young Tile Theorem:

If with every point x of E, there is associated a sequence of spheres, whose radii converge to zero, then for every e it is possible to find a denumerable set of these spheres E_n, such that

$$\sum \text{meas}\, C_n < \overline{\text{meas}}\, E + e$$

and every point of E is interior to at least one of the spheres.

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* Monatshefte für Mathematik und Physik, vol. 27 (1916), pp. 189–190.

This is a consequence of the Vitali Theorem and the result that under the hypothesis of the theorem, there exists a denumerable set of spheres covering E and a constant k (which depends upon the dimension of the space) such that

$$\sum \text{meas}\, C_n < k(\overline{\text{meas}}\, E + e).$$

Essentially the point is that under the given hypothesis it is possible for every e to cover a set of measure zero by a denumerable set of the given spheres, the sum of whose measures is less than the given e.

It is obvious that in the Rademacher theorem, the spheres can be replaced by cubes having the points x as center. To further extensions there are limitations. J. Splaya-Neumann* has shown that it is necessary that the points x be the centers of the spheres by giving an example of a plane closed set of measure zero, such that the sum of the measures of the covering circles is always greater than or equal to unity.†

For the linear interval, J. C. Burkill‡ has given an exact covering theorem based on intervals, an extension of the Vitali Theorem. By observing that as a result of the metric density theorem the finite number of intervals of the Vitali Theorem can be chosen so that each of the complementary intervals contains a point of the given set, and combining this with the Borel Theorem he obtains the following slightly complicated result:

If $\mathfrak{F}$ *is a given family of intervals I satisfying the hypotheses of the Vitali Theorem relative to a closed set E, and* $\mathfrak{G}$ *a family of intervals J such that for every x of E all intervals in a certain vicinity of x belong to* $\mathfrak{G}$, *then there exists an interval K which*

* FUNDAMENTA MATHEMATICAE, vol. 5 (1924), pp. 329–30.

† Cf. also K. Menger, WIENER BERICHTE, vol. 133 IIa (1924), pp. 425–7; and R. L. Moore, loc. cit., pp. 464–5, where examples based on intervals, rectangles, and squares are given, which are not centered relative to the points of association. Moore's example, however, is not with respect to a set of measure zero, as he claims.

‡ FUNDAMENTA MATHEMATICAE, vol. 5 (1924), pp. 322–4.

for every e is completely covered without overlapping by a finite number of intervals from $\mathfrak{F}$ *and* $\mathfrak{G}$ *fulfilling the conditions*

$$\text{meas}(\Sigma I_n - (\Sigma I_n)EK) < e, \quad \textit{and} \quad \text{meas}(EK - (\Sigma I_n)EK) < e.$$

We note finally that in the Vitali Theorem, the sets I of the family $\mathfrak{F}$ may be replaced by measurable sets, but then the condition that the subfamily consist of non-overlapping sets must be dropped.*

II. THE BOREL THEOREM IN GENERAL SPACES.

Probably no theorem of analysis has contributed more towards the analysis of general spaces than the Borel Theorem. The attempts to derive the theorem in increasingly general situations has led to interesting new properties and characterizations of spaces.

8. *Metric Space.* The first and simplest general space to which the Borel Theorem was extended is now generally called a metric space. The definition of the space and the proof of the theorem in this space were made by Fréchet in his Paris thesis.† A metric space $\mathfrak{D}$ consists of a set of general elements x. It is postulated that for every pair of elements x_1 and x_2 of the space there exists a positive real number $\delta(x_1, x_2)$ called distance and subject to the conditions

(1) $\delta(x_1, x_2) = \delta(x_2, x_1)$ for every x_1 and x_2,

(2) $\delta(x_1, x_2) = 0$ if and only if x_1 and x_2 are identical,

(3) $\delta(x_1, x_2) \leqq \delta(x_1, x_3) + \delta(x_3, x_2)$ for every x_1, x_2 and x_3.

A sequence $\{x_n\}$ is said to have as *limit* the element x if $\lim_n \delta(x_n, x) = 0$. A set E in the space $\mathfrak{D}$ is said to have x as *limiting element* if there exists a sequence of distinct elements $\{x_n\}$ extracted from E having x as limit. *Derived sets*, *closed sets*, and *perfect sets* are defined in the usual way,

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* See Lebesgue, loc. cit., p. 394; and Rademacher, loc. cit., pp. 191–2.

† *Sur quelques points du calcul fonctionnel*, PALERMO RENDICONTI vol. 22 (1906), pp. 1–72. Fréchet's results were stated with respect to what seemed to be a slightly more general situation, the equivalence with the above metric space being shown by E. W. Chittenden, TRANSACTIONS OF THIS SOCIETY, vol. 18 (1917), pp. 161–6.

and the notation E' is used for the derived set of E. In such a space it seems natural to define a *sphere* center x_0 and radius r as the totality of points of $\mathfrak{D}$ satisfying the condition

$$\delta(x, x_0) \leqq r,$$

also to speak of a sphere of radius e as a *vicinity* of its center. The notion *x interior to the set E* can be defined in the following equivalent ways:

(a) there exists a sphere having x as center all of whose points belong to E, or

(b) x belongs to E and is not a limiting element of any set consisting of points all of which do not belong to E. From the properties of δ it follows that all points x such that $\delta(x, x_0) < r$ are interior to the sphere center x_0 and radius r.

In a general metric space the Weierstrass-Bolzano Theorem is not a consequence of the boundedness of a set. Instead we have the property *compact*, a set E being compact, if every infinite subset has a limiting element. If every infinite subset of E has a limiting element in E, we shall call E *self-compact*,* which concept in a metric space is equivalent to compact and closed.

As in the linear case, we shall say that a family $\mathfrak{F}$ of sets I chosen from a space $\mathfrak{D}$ covers the set E if every point of E is interior to some set I of $\mathfrak{F}$. Then the Borel Theorem can be stated:

Any self-compact set E which is covered by a family $\mathfrak{F}$, can be covered by a finite subfamily of $\mathfrak{F}$.

It may occasionally be useful to refer to the property expressed in this theorem as the *Borel property*, i.e., E has the Borel property if from any family covering E, a finite subfamily covering E can be selected.

Fréchet proved the theorem first for the case in which the family is denumerable. The process is very much the same as that given in §2(d), for the linear case. The family is arranged in sequential order $I_1, \cdots, I_n, \cdots$, and by a step-by-step process replaced by an array $I_1, \cdots, I_{nm}, \cdots$,

* Due to E. W. Chittenden, this BULLETIN, vol. 21 (1915), p. 18.

in which each set contains at least one element of E not interior to the preceding set. If this sequence is infinite, we get a sequence of distinct elements of E, $x_1, \cdots, x_m, \cdots$, where x_m is interior to I_{nm} but not to any preceding set. By the self-compactness of E, this sequence has a limiting element x_0 of E. Now x_0 is interior to some set I_n and consequently I_n contains a sphere having x_0 as center, and hence an infinite number of the sequence x_m as interior points. This leads to a contradiction with the method of selecting I_{nm} and x_m.

We call attention to the fact that the proof utilizes in particular two ideas, (a) the self-compactness of E, and (b) the fact that if x is interior to E, and a limiting element of E_1 then E contains as interior elements an infinite number of elements of E_1.

For the general Borel Theorem, Fréchet* originally postulated further properties of the space, viz., that the space $\mathfrak{D}$ is *separable*, i.e., can be considered as the derived set of a denumerable set of its elements. It was shown later that any compact set E of a space $\mathfrak{D}$ has the same property, i.e., for any compact set E, there exists a denumerable subset E_0 such that E is contained in E_0+E_0', thus removing the restriction of separability for the space $\mathfrak{D}$.†

The proof of the Borel Theorem depends upon the following lemma.

LEMMA.‡ *If every point x of a compact set E is the center of a sphere of radius $r(x)$, and if there exists an e such that for all x of E*

$$r(x) > e > 0,$$

then all points of E are interior to a finite number of these spheres.

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* Loc cit., pp. 25–27.

† See T. H. Hildebrandt, AMERICAN JOURNAL, vol. 34 (1912), pp. 278–281; W. Gross, WIENER BERICHTE, vol. 123 IIa(1914), pp. 809–812; Fréchet, BULLETIN DE LA SOCIÉTÉ DE FRANCE, vol. 45 (1917), pp. 1–8.

‡ See Hahn, *Reelle Funktionen*, 1921, pp. 89–93. See also Urysohn, FUNDAMENTA MATHEMATICAE, vol. 7 (1925), pp. 46–48, where the role of the axiom of choice in the proof of the Borel Theorem is emphasized.

Let $S(x_1)$ be the sphere about any point x_1 of E, and x_n any point not interior to $S(x_1)$, $S(x_2)$, $\cdots$, $S(x_{n-1})$. Then on account of the fact that $\delta(x_n, x_m) > e$ for every n and m and E is compact, the sequence x_n is finite.

With this lemma, the Borel Theorem in a metric space can be proved in two ways. Either (a) following the method of §2(c) on the linear interval, let $r(x)$ be the least upper bound of the radii of the spheres of center x interior to some set I of the given family. The self-compactness of the family yields immediately that the $r(x)$ have a positive lower bound, and the lemma then suggests a method for selecting the finite subfamily from $\mathfrak{F}$.

Or (b) from the lemma we conclude that for every n the points of a compact set E are interior to a finite number of spheres of radii $1/n$. For every n then we retain the spheres which are interior to some set I of $\mathfrak{F}$. We obtain in that way a denumerable family of spheres, covering E, and hence by the denumerable-to-finite Borel Theorem, we can select a finite subfamily of spheres, which in turn defines a finite subfamily of $\mathfrak{F}$ covering E.

This last proof contains practically the proof of the result that *any compact set E is separable.* For the centers of the spheres of radius $1/n$ having E as interior points will be a denumerable set E_0 having the property that E belongs to E_0+E_0'.

Also there is present a special case of the Lindelöf Theorem in a metric space, viz.

If in a space $\mathfrak{D}$ E is any separable set covered by a family $\mathfrak{F}$ of sets I, then it is covered by a denumerable subfamily.

Obviously the reason why the Lindelöf theorem holds in linear or n-dimensional space is because these spaces are separable. The method of proof is entirely similar to these special instances. Let $E_0=[x_n]$ be a denumerable subset of E such that E belongs to E_0+E_0'. Consider the denumerable family of spheres, center x_n, and rational radii interior to some I of $\mathfrak{F}$. They will cover E and set up a one to one correspondence with a denumerable subfamily of $\mathfrak{F}$.

The condition that E is self-compact is necessary for the validity of the Borel Theorem. For if E is not self-compact, then there exists a denumerable set of elements $[x_n]$ of E without a limiting element in E. To every point x of E there corresponds then a sphere containing at most one point of $[x_n]$. These spheres form a family covering E, but every finite subfamily contains only a finite number of points of $[x_n]$ and hence does not cover E.

Similarly the condition that E be separable is also necessary for the Lindelöf Theorem. Instead of proving this directly we relate this theorem and separability to a third property suggested in Lindelöf's paper. If we define an *element of condensation* of a set E, as a limiting point of E which remains a limiting point after the removal of any denumerable set from E, and a set as *self-condensed* if every non-denumerable set chosen from E has at least one element of condensation in E, then we have the equivalence of the following three properties in a metric space $\mathfrak{D}$:*

(A) *The set E is separable.*

(B) *The set E is self-condensed.*

(C) *The set E has the Lindelöf property*, i. e., if $\mathfrak{F}$ is any family of sets covering E, then a denumerable sub-family of $\mathfrak{F}$ covers E.

We have already shown that (A) implies (C). To show that (C) implies (B) follows the lines of the converse of the Borel Theorem above; the assumption of a non-denumerable set E_0 without an element of condensation in E yields via (C) a denumerable set of spheres each of which contains only a denumerable set of elements of E_0. To show finally that (B) implies (A), we show first that if E is condensed and if each point of E is the center of a sphere of radius greater than e, then all the points of E are interior to a denumerable

* See Gross, loc. cit., pp. 805–12; Fréchet, ANNALES DE L'ÉCOLE NORMALE, (3), vol. 38 (1921), pp. 349–356.

subset of these spheres. Allowing e to take on successively the values $1/n$, we get a denumerable set of centers of spheres which is the desired denumerable subclass of E.

In so far as measure has not yet been effectively connected with a general metric space, it is not possible to give generalizations of the Vitali Theorem. However K. Menger* has given some results which are comparable to the Vitali Theorem.

We define the *diameter* of a set I as the diameter of the minimum sphere containing I. Then Menger is interested in the question: What properties of a set E in a metric space and what properties of a family $\mathfrak{F}$ of sets I such that every point of E is interior to a subfamily of sets I whose diameters converge to zero, are sufficient to make possible the selection of a finite or denumerable subfamily of $\mathfrak{F}$ of sets I whose diameters approach zero, and which covers E.

If $d(I)$ is the diameter of I and $d(I, x)$ is the least upper bound of the diameters of spheres center x contained in $I(x)$, then Menger's principal result is that the selection desired is possible provided

(a) E is the sum of a denumerable set of compact sets;

(b) if $I(x)$ are the sets of $\mathfrak{F}$ associated with x, then for every x the condition that $d(I)$ approach zero is a consequence of the fact that $d(I, x)$ approaches zero. This latter condition is fulfilled in case there exists a positive-valued function $f(x)$ on E such that for each I associated with x

$$d(I) < f(x) \cdot d(I, x) .$$

The proof is made first for a compact set E. If E_n is the subset of E for which there exists an I such that $d(I, x) > 1/n$ then E_n is interior to a finite number of these sets I. As a consequence E can be covered by a finite or denumerable family of sets I_n from $\mathfrak{F}$, which if denumerable has the property that there exists a point x_n of I_n such that $d(I_n, x_n)$ approaches zero, so that $d(I_n)$ converges to zero

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* *Einige Überdeckungssaetze der Punktmengenlehre*, WIENER BERICHTE, vol. 133IIa (1924), pp. 421–444.

as desired. The extension to the denumerable set of compact sets is obvious.

In addition to having contact with the Vitali Theorem, the results of Menger are also related to Young's Tile Theorem (§6) and R. L. Moore's reduction theorem (§5). The conditions of Menger are sufficient. No doubt further interesting results can be obtained by considering necessary conditions.

9. *The Borel Theorem in a Space $\mathfrak{L}$ with Limit of Sequence defined.* Fréchet's thesis besides considering metric spaces also postulated a more fundamental space $\mathfrak{L}$, that in which limit of a sequence is defined. Limit is subject to three conditions: (1) the limit of a sequence is unique, (2) the limit of a sequence consisting of the same element repeated is this element, and (3) any subsequence of a sequence having a limit has the same limit. Obviously limiting element, interiority, and the other concepts can be defined in the main as suggested for metric spaces.

The first statement of a Borel Theorem in a space $\mathfrak{L}$ is due to E. R. Hedrick.* By analyzing the proof for the Borel Theorem in the denumerable-to-finite case, he observed that it could be effected provided the space $\mathfrak{L}$ had the following property† (called by Fréchet the Hedrick property):

(H) *If x is interior to a set E, then an infinite number of elements of any sequence having x as limit are interior to E.*

He found that this property was a consequence of the simpler property:

(S) *The derived class of any class is closed.*

For suppose x is interior to E and the limit of a sequence of distinct elements x_n, the result being obvious if the elements are not distinct. We show that there exists an n_0

* TRANSACTIONS OF THIS SOCIETY, vol. 12 (1911), pp. 285–7.

† Fréchet's statement (cf. ANNALES DE L'ÉCOLE NORMAL, (3), vol. 38 (1921), p. 348) of this property is: If x is interior to E and a limiting element of F, then x is a limiting point of a subset F_0 of F consisting entirely of elements interior to E. In an $\mathfrak{L}$ space this statement is equivalent to the one above.

such that for $n > n_0$, x_n is interior to E. The assumption of the contrary gives rise to a sub-sequence $\{x_{n_m}\}$ no point of which is interior to E, i. e., each member of the sub-sequence is the limiting element of a sequence of elements $\{x_{n_mk}\}$ not members of E. If we let E_0 be the class of elements $[x_{n_mk}]$, then E_0' contains the sequence $\{x_{n_m}\}$ and consequently the point x, by the property S. The element x would then be a limiting element of E_0 which would contradict the interiority condition of x to E.

It is now fairly obvious that the Borel Theorem is valid in the form

If the space $\mathfrak{L}$ has the property S *then any self-compact set E covered by a denumerable family $\mathfrak{F}$ of sets I, is covered by a finite sub-family of $\mathfrak{F}$*, i. e., any self-compact set E has the denumerable-to-finite Borel property.

The condition that E be self-compact is necessary in any space $\mathfrak{L}$.* For suppose $\{x_n\}$ is a sequence chosen from E not having a limiting element in E. Let I_m be the set obtained from the given space by deleting the elements x_n for $n > m$. Then obviously the set E is covered by the family $\mathfrak{F}$ of I_m, since no element of E is a limiting element of $\{x_n\}$, but no finite subfamily of $\mathfrak{F}$ contains all points of the sequence $\{x_n\}$.

Further the property S and so H is a necessary property in a space $\mathfrak{L}$ for the Borel Theorem in this form: if E is self-compact then E has the denumerable-to-finite Borel property.† Let if possible the set E be such that E' is not closed. Then there exists a sequence $\{x_n'\}$ of elements of E' with a limit x'' not belonging to E'. Let the elements $\{x_{mn}\}$ of E be such that x_n' is a limiting element of x_{mn} for $m=1, 2, \cdots$. Consider the family $\mathfrak{F}$ consisting of (a) I_0, the set remaining after removing the elements x_{mn} from the fundamental space, and (b) I_n the set remaining after

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* Cf. E. W. Chittenden, *The converse of the Heine-Borel theorem in a Riesz domain*, this BULLETIN, vol. 21 (1915), pp. 179–183.

† Cf. Fréchet, BULLETIN DE LA SOCIÉTÉ DE FRANCE, vol. 45 (1917), pp. 1–8; Chittenden, this BULLETIN, vol. 25 (1918), pp. 60–66.

removing all members of the sequence $\{x_m'\}$ excepting x_n' from the fundamental space. Then the self-compact set E_0 consisting of the sequence $\{x_n'\}$ and x'' is covered by the family $\mathfrak{F}$ but by no finite subfamily of this family.

We note that *in a space* $\mathfrak{L}$ *there is then equivalence between these three properties*:

(S) *The derived class of every class E is closed.*

(H) *If x is interior to E, and is the limiting element of a set E_1 then an infinite subset of E_1 is interior to E.*

(B) *If E is self-compact, it has the denumerable-to-finite Borel property.*

The problem of determining conditions under which the any-to-finite Borel Theorem is valid in a space $\mathfrak{L}$ remained unsolved for some time. Obviously the methods of metric space could not be used. A method of attack is suggested by the proof in the case just treated, viz. to utilize the theory of transfinite ordinals. The first solution of the problem was given by R. L. Moore.* He calls a *monotonic family of classes G*, a family such that for each pair G_1 and G_2 of the family, one contains the other. He then defines the concept called by Fréchet *perfectly compact.* The set E is perfectly compact in case every infinite monotonic family of sets chosen from E or the family of their derived sets has a common element. That a perfectly compact set is compact is obvious from a consideration of the monotonic family where G_m consists of the elements of the sequence $\{x_n\}$ for $n>m$.† The relationship of the property "perfectly compact" to the theorem "any monotonic sequence of closed compact sets has a common element" will appear later. Moore's result is as follows.

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* See PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES, vol. 5 (1919), pp. 206–210.

† Fréchet (ANNALES DE L'ÉCOLE NORMALE, (3), vol. 38, pp. 334–6) shows that in a metric space every compact set is also perfectly compact. This can also be deduced from the converse of the any-to-finite Borel Theorem in a metric space and in an $\mathfrak{L}$ space.

In a space $\mathfrak{L}$ with the property S, *a necessary and sufficient condition that E have the any-to-finite Borel property is that E be perfectly self-compact.*

Suppose there is given a family $\mathfrak{F}$ of sets I which covers E. Well-order this family, and then by a step-by-step process reduce the family so that each I contains as interior point at least one element of E not interior to any of the preceding sets in the array. Let the members of the resulting family be $I_1, I_2, \cdots, I_\alpha, \cdots$, and let x_α be interior to I_α but not to I_β for $\beta > \alpha$. Let E_α be the set of points x_β for $\beta > \alpha$. Then the family consisting of the E_α is a monotonic family from E, and since the E_α do not have a common point, either the family is finite or the derivatives E_α have a common point x'. In the latter case there exists a γ such that x' is interior to I_γ, and since x' is a limiting element of E_γ, by the property H it follows that I_γ contains interior points of the set of x_α for ordinals greater than γ, contrary to the definition of the x_α. Hence the number of members of the family of E_α is finite.

For the proof of the converse, let E be a set of the space $\mathfrak{L}$ having the any-to-finite Borel property. Then as shown above E is self-compact, i. e. compact and closed. Let G be any infinite monotonic family of sets E_α drawn from E. Then since E is closed E_α' will belong to E. Moreover the property S insures that the sets $\overline{E}_\alpha = E_\alpha + E_\alpha'$ are closed. Let the sets I_α of the family $\mathfrak{F}$ consist of the points remaining after deleting the elements of $\overline{E}_\alpha$ from the fundamental set. If the sets $\overline{E}_\alpha$ have no common element, it follows that E will be covered by the family $\mathfrak{F}$, but a finite subfamily of $\mathfrak{F}$ will not cover E since the sets $\overline{E}_\alpha$ contain elements for each α.

Another solution of this question of the Borel any-to-finite Theorem in a space $\mathfrak{L}$ was given in 1923 by Kuratowski and Sierpinski.* In so far, however, as their result is practically stated in a space in which vicinities are defined we shall postpone consideration of it to a later section of this paper.

* *Le théorème de Borel-Lebesgue dans la théorie des ensembles abstraits*, Fundamenta Mathematicae, vol. 2 (1921), pp. 172–8.

10. *Vicinity Spaces* $\mathfrak{V}$. *Hausdorff Form of the Borel Theorem.* The first suggestions of sets or vicinities as the basis for consideration in a general space are to be found in the paper of Hedrick.* Another development based to some extent on the Riesz† postulates for limiting element was given by R. E. Root.‡ About the same time, Hausdorff in his book on Mengenlehre developed systematically the point set theory in a vicinity space. Later, in 1918, Fréchet§ gave an independent development of the same type of space showing in particular the relationship between the space characterized by the Riesz postulates and a space based upon vicinities. The Hausdorff postulates have come to be accepted as a satisfactory basis, and a space based on them is usually called a topologic space. The postulates are as follows:

I. To every point x there corresponds a family of sets $V(x)$, chosen from the given space, and containing x.

II. If $V_1(x)$ and $V_2(x)$ are vicinities of x, then there exists a common subvicinity $V_3(x)$.

III. For every pair of points x_1 and x_2, there exist vicinities $V_1(x_1)$ and $V_2(x_2)$ without common elements.

IV. If x_2 belongs to $V_1(x_1)$ then there exists a $V_2(x_2)$ contained in $V_1(x_1)$.

In a vicinity space *limiting element* of a class E can be defined either (a) x is a limiting element of E if every $V(x)$ contains at least one point other than x, or (b) x is a limiting element of E if every $V(x)$ contains an infinity of

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* Loc. cit., p. 289; Fréchet, TRANSACTIONS OF THIS SOCIETY, vol. 14 (1913), pp. 320–4, showed that the space postulated by Hedrick was a metric space.

† See ATTI DEL IV CONGRESSO INTERNAZIONALE (Roma) 1909, vol. 2, pp. 18–22.

‡ Cf. TRANSACTIONS OF THIS SOCIETY, vol. 15 (1914), pp. 51–70.

§ BULLETIN DES SCIENCES MATHÉMATIQUES, (2), vol. 42 (1918), pp. 138–156; called Fréchet I in the sequel. Fréchet considers a type of space that he has called "espace accessible," which is equivalent to a vicinity space subject to postulates similar to those of Hausdorff, IV and especially III being replaced by weaker ones.

elements. In a $\mathfrak{V}$ space satisfying conditions I, II, and III these two definitions are equivalent.

More generally we can define *x is a limiting element of E of power* μ, if every vicinity of x contains a subset of power μ. Finally x is called a *complete limiting element of E* if every $V(x)$ contains a subset of E of the same power as E.

It is obvious that we can obtain an $\mathfrak{L}$ space in a $\mathfrak{V}$ space by assuming that $\lim_n x_n = x$ is defined "for every $V(x)$ there exists an n_0 such that if $n > n_0$ then x_n is contained in $V(x)$." But limiting element based on the sequence notion of this $\mathfrak{L}$ space need not agree with the limiting element of the given $\mathfrak{V}$ space, unless the $\mathfrak{V}$ space is subjected to additional conditions.* On the other hand given an $\mathfrak{L}$ space it is possible to define it as a $\mathfrak{V}$ space in which limiting elements are the same.†

The notion *interiority* can be defined in different ways, equivalent if we are in a Hausdorff $\mathfrak{V}$ space (i. e. subject to conditions I, II, III and IV)‡: *x is interior to E* if x belongs to E and either (a) every set E_1 having x as a limiting element, contains at least one element other than x of E, or (b) every set E_1 having x as limiting element contains an infinity of elements of E, or (c) there exists a $V(x)$ containing only elements of E.

Obviously x is interior to every $V(x)$.

An *open set* or *region* is a set containing only interior points. On account of condition IV every $V(x)$ is an open set. We note that the sum of two open sets is open, also the complementary set of a closed set is open.

Finally we note that on account of condition IV, a Hausdorff $\mathfrak{V}$ space has the property S.

The outstanding difference in Hausdorff's statement of the Borel Theorem from that stated by Fréchet is that the

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* Cf. Root, loc. cit., pp. 67–71; Fréchet, I, p. 148.

† Cf. Fréchet, I, pp. 140–148.

‡ Limiting element may be of power 2 or $\aleph_0$. Definition (c) is perhaps most satisfactory in so far as limiting element does not enter directly.

family $\mathfrak{F}$ of covering sets consists of open sets, which introduces an element of simplicity, since belonging to an open set is equivalent to being interior. The most elegant form of the Borel Theorem in a Hausdorff $\mathfrak{V}$ space has been given by P. Alexandroff and P. Urysohn,* by calling attention to the following equivalences:

THEOREM I. *The following properties of a set E in a Hausdorff space are equivalent:*

A_0. *E is self-compact.*

A_1. *Every denumerable subset of E has a complete limiting element in E.*

B. *If $\mathfrak{F}$ is a denumerably infinite monotonic family of closed sets F_n such that each F_nE contains at least one element, then the sets F_nE have a common element.*

C. *If E is covered by a denumerable family $\mathfrak{G}$ of open sets G, then E is covered by a finite subfamily of $\mathfrak{G}$.*

It is obvious that A_0 and A_1 are equivalent. The equivalence of B and C is a matter of taking complementaries, products and sums. Assume B, and let $\mathfrak{G}=[G_n]$. Then the sets $\sum_1^m G_n$ are open sets, and the sets $\mathfrak{V}-\sum_1^m G_n$ closed, as a matter of fact form a monotonic family. If the sets $E(\mathfrak{V}-\sum_1^m G_n)$ contain an element for every m, then by B they have a common element, i. e. the family $\mathfrak{G}$ does not cover E. Conversely, assume C and a monotonic family of closed sets $[F_n]$ such that each F_nE contains a point. Then the $\mathfrak{V}-F_n$ are open sets. If the F_nE do not have a common point, then the sets $\mathfrak{V}-F_n$ will contain all points of E. The finiteness of the equivalent set chosen from the $\mathfrak{V}-F_n$ leads to a contradiction.

The fact that A_1 implies B is a well known result. The denumerable family F_nE leads to a set E_0, of points x_n where x_n belongs to F_nE. The complete limiting element of E_0 in E is also a limiting element for each F_nE, and hence in each member of the family.

* Cf. MATHEMATISCHE ANNALEN, vol. 92 (1924), pp. 258–60.

To complete the equivalence we show that C implies A_0. If possible let E_0 be a denumerable set without a limiting element in E. Then $\bar{E}_0=E_0+E_0'$ will be closed and have no limiting elements in E. Then if x_n is any element of E_0, the sets

$$G_n=\mathfrak{D}-\bar{E}_0+x_n$$

will be open and will contain all points of E, but a finite subfamily of the G_n will not.

More generally we have the following theorem:

THEOREM II. *The following properties of E in a Hausdorff $\mathfrak{B}$ space are equivalent:*

A. *Every infinite subset of E has a complete limiting element belonging to E.*

B. *If $\mathfrak{F}$ is any well-ordered monotonic decreasing family of closed sets $\mathfrak{F}$ such that for each α, $F_\alpha E$ contains an element, then the sets $F_\alpha E$ have a common element.*

C. *If E is contained in a family $\mathfrak{G}$ of open sets G, then it is contained in a finite subfamily.*

The proof of the equivalence of B and C follows the lines of the corresponding proof for Theorem I.

Assume A, and let $\mathfrak{G}$ be a family of open sets G.* Let μ be the power such that any subfamily of $\mathfrak{G}$ of power less than μ does not contain E, but there are subfamilies of power μ containing E. Let $\mathfrak{G}_0$ be a subfamily of $\mathfrak{G}$ of power μ containing E. Let Ω be the least ordinal of power μ. Then we can well-order $\mathfrak{G}_0$ in the form $G_1, G_2, \cdots, G_\alpha, \cdots$ such that $\alpha<\Omega$ and by possible deletion assume that each G_α contains at least one point x_α of E not in any preceding set. Consider the set $E_0=[x_\alpha]$. By the property A if E_0 is infinite, E_0 has a complete limiting element x, belonging to E, and consequently to some member G_β of the family $\mathfrak{G}_0$. Consequently every vicinity of x and so also G_β will contain a subset of E_0 of power μ. Since the set of elements x_α for $\alpha<\beta$ is of power less than μ it follows that G_β contains

* This method of proof follows the lines suggested by Kuratowski and Sierpinski, loc. cit., pp. 174–5.

points of E_0 of index greater than β, contrary to the method of choice of the x_α. Hence the set E_0 is finite.

To complete the equivalence we show that C implies A. For suppose E_0 is an infinite subset of E not having a complete limiting point in E. Then for every x of E there exists a vicinity $V(x)$ such that the power of $V(x)E_0$ is less than that of E_0. These vicinities being open sets, constitute a family of open sets containing E. But obviously a finite subfamily cannot contain all points of E_0.

Theorem I suggests an extension in which the word "denumerable" is replaced by "power less than or equal to μ," in the properties A_1, B and C, the proof being similar. Theorem II suggests the following extension.

THEOREM III. *The following properties of a class E in a Hausdorff $\mathfrak{B}$ space are equivalent*:

A. *Every infinite set of power μ has a complete limiting element in E.*

B. *If $\mathfrak{F}$ is a well-ordered monotonic decreasing family of closed sets F, of power $\geqq \mu$, such that for each α, $F_\alpha E$ contains at least one element, then the sets $F_\alpha E$ have a common element.*

C. *If E is contained in a family $\mathfrak{G}$ of open sets G, the power of $\mathfrak{G}$ being $\geqq \mu$, then E is covered by a subfamily of $\mathfrak{G}$ of power $\leqq \mu$.*

This theorem contains among others the Lindelöf Theorem as a special case. The proof is similar to that of Theorem II. It is interesting that in the general space the Lindelöf Theorem and the Borel Theorem seem to join hands, a fact not to be foreseen by a consideration of n-dimensional space.

Alexandroff and Urysohn call a set E satisfying the conditions of Theorem II *bicompact*, because it is a meeting place of the generalization of Theorem I and of Theorem III. It is obvious however that property B is a special case of the *perfectly compact* property. Perhaps *completely compact* would be a better term.

It remains to consider to what extent the Hausdorff postulates on the $\mathfrak{V}$ space are needed. An analysis of the proofs shows that in any space in which closed and open sets are complementary the properties B and C are equivalent.*

In Theorem I, A_0 and A_1 are equivalent, and imply B and C if the $\mathfrak{V}$ space is subject only to conditions I, provided "limiting element" is the limiting element of power $\aleph_0$ (i. e. definition (b)). The converse that A_0 and A_1 follow from B and C requires condition IV, making $V(x)$ an open set, which condition is practically the property S: the derived set of a set is closed. If "limiting element" is of power 2, (i. e. definition (a)), then B and C follow from A_1, but A_0 and not A_1 follows from B and C under additional postulate IV.

Theorems II and III are true under a space satisfying postulates I and IV, the latter being required in the proof of the result "C implies A," for instance.

11. *General Borel Theorem in a $\mathfrak{V}$ Space.* We have pointed out that the Hausdorff statement of the Borel Theorem is based on families of open sets. It seems desirable to consider briefly what happens in case we are dealing with families of arbitrary sets, the deciding covering property being then interiority. At the same time, the attempt is to reduce the properties of the $\mathfrak{V}$ space to a minimum.

For most of this section, we shall assume considerations based on a $\mathfrak{V}$ space subject to condition I of Hausdorff. *Limiting element* is of power 2, i. e., x is a limiting element of E in case every vicinity of x contains a point of E other than x. Further *x is interior to E* if x belongs to E and every set having x as limiting element has a point other than x in common with E, or if E contains a vicinity of x. Compactness, derived sets, limiting points of power μ, complete limiting points are defined as above. We use finally a new concept: x is a *complete interior limiting point*†

* Cf. Saks, FUNDAMENTA MATHEMATICAE, vol. 2 (1921), pp. 1–3.

† Chittenden (this BULLETIN, vol. 30 (1924), p. 556, referred to as Chittenden I in the sequel), calls such a point a hypernuclear point.

of E if every vicinity of x contains as *interior* points a set of the same power as E. The definition of interior limiting point of power μ is then obvious.

As in the previous section it seems proper to consider the following properties of a set E in this space as being connected with the denumerable-to-finite Borel Theorem:

A_0. E is self-compact.

A_1. Every denumerable subset of E has a complete limiting point in E.

A_2. Every denumerable subset of E has a complete interior limiting point in E.

B. If $\mathfrak{F}$ is a monotonic denumerable family of sets F chosen from E, either the sets F or their derived sets have a common element in E.

C. If E is covered by a denumerable family $\mathfrak{G}$ of sets G, then it is covered by a finite sub-family of $\mathfrak{G}$.

We obviously have that A_2 implies A_1 implies A_0.

The statement "A_2 implies C" is a form of the Borel Theorem whose proof is obvious. The converse C implies A_0 can be proved by assuming if possible E not self-compact. Then there exists a denumerable set E_0 of elements of E without a limiting element in E. Then for each point y_n of E_0 there exists a $V(y_n)$ containing only the point y_n of E, and for each point x of E not a point of E_0 a vicinity $V(x)$ containing no points of E_0. Then the sets

$$G_0 = \sum V(x), \quad G_n = V_n(y_n)$$

cover E but a finite subfamily does not.

The proof of the fact that B follows from A_1 can be modelled after the more general result given below. The converse is is obvious.

To obtain further results it seems necessary to add additional hypotheses on the fundamental space.

The assumption that $\mathfrak{V}$ satisfies the condition IV of Hausdorff, or has the property S, that derived classes are closed, gives a Hedrick property which can be stated as follows.

(H) *If x is interior to E and a limiting element of E_0 of power μ then E_0E contains a set of interior elements, whose power is μ.*

If derived classes are closed, then according to Fréchet* every vicinity $V(x)$ of x contains a subvicinity $V_0(x)$ all of whose elements are interior to $V(x)$. Now if x is interior to E, then there exists a vicinity $V(x)$ of x, which contains only points of E. The subvicinity $V_0(x)$ of V then defines a subset of E_0 of power μ all points of which are interior to E.

If $\mathfrak{V}$ has the property S and consequently H, then we can state the equivalence of B and C. The proof of this equivalence follows the lines of proof of Moore's form of the Borel Theorem in an $\mathfrak{L}$ space.†

The other possible equivalences seem to be linked up with the fact that in the general $\mathfrak{V}$ space with limiting element as defined, a finite class may have a limiting element. The conditions II and III of Hausdorff are sufficient to guarantee the contrary. Under these conditions it is possible to prove that C implies A_2 and A_0 implies A_1. The proof of the former of these statements follows the lines of the proof of the fact that C implies A_0 given above. The latter is obvious. We have as a consequence the following theorem.

THEOREM I. *If the $\mathfrak{V}$ space satisfies condition* IV *of Hausdorff, then properties* A_1, B *and* C *are equivalent. If the space is a Hausdorff $\mathfrak{V}$ space, then all the properties* A_0, A_1, A_2, B, *and* C *are equivalent.*

The extension of these results to the case where the word "denumerable" is replaced by "power less than or equal to μ" in the properties A_1, A_2, B, and C is obvious.

Theorem II of § 10 suggests the consideration of the following properties:

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* Cf. Fréchet, I, p. 145.

† Chittenden (I, p. 519) has shown that in a general $\mathfrak{V}$ space the property S is not a consequence of the Borel Theorem in the form "A_0 implies C."

A_1'. Every subset of E has a complete limiting element in E.

A_2'. Every subset of E has a complete interior limiting element in E.

B′. E is perfectly self-compact, i. e., if $\mathfrak{F}$ is any monotonic family of sets F chosen from E then either the sets F or their derived sets have a common element in E.

C′. E has the Borel any-to-finite property, i. e., if E is covered by a family $\mathfrak{G}$ of sets G then E is covered by a finite subfamily of $\mathfrak{G}$.

Chittenden* has stated the following theorem.

THEOREM II. A_1 *is equivalent to* B′ *and* A_2' *is equivalent to* C′.

We prove first that A_1' implies B′. Suppose $\mathfrak{F}$ a monotonic family of sets F chosen from E. Let H be a set of elements of E such that every element of H is in some F and every F contains an element of H. Well-order H, x_α corresponding to the ordinal α. Then we determine a well-ordered subfamily of $\mathfrak{F}$ and a subset H_0 of H by the requirement that F_α contain no element of H with ordinal $\gamma<\beta_\alpha$ and x_{β_α} be the first element of H common to all F_γ for $\gamma \leqq \alpha$. Then for every F of $\mathfrak{F}$ there exists an α such that F contains F_α. For F contains an element x_γ of H, and for some α, x_γ will not belong to F_α, so that by the monotonic character of $\mathfrak{F}$, F contains F_α. From this it follows that if the derived sets F_α' have a common element, the same will be true of the sets F', i.e. if the theorem holds for a well-ordered monotonic family, it holds for any monotonic family.

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* I, pp. 514–8. Very recently, since this paper was in type, Professor Chittenden has called my attention to the fact that the proofs of the first part of this theorem contained in his paper are not correct. Correct proofs by Sierpinski, who first noted the error, will appear in the next issue of this BULLETIN. The proof that is given here of the fact that A_1' implies B' is a modification of Chittenden's proof, obtained before I was aware of an error in his work.

Let $F_1, \cdots, F_\alpha, \cdots$ be the elements of $\mathfrak{F}$ in order. Since every well-ordered set without final element is cofinal with a regular ordinal number,* it is obviously sufficient to prove the theorem for the case where every ordinal α precedes Ω the least transfinite ordinal corresponding to μ the power of $\mathfrak{F}$. Let x_α belong to F_α but not to $F_{\alpha+1}$. Then from the assumption concerning E it follows that the set H of elements x_α will have a complete limiting element x'. Every vicinity of x' contains a set of H of power μ, i.e. for every α an element x_β with $\beta>\alpha$, and consequently an element of F_α. Hence x' is common to the sets F_α'.

For the proof of the converse, that B' implies A_1', we refer the reader to the paper by Sierpinski, which will appear in the next issue of this BULLETIN.

The proof of the equivalence of A_2' and C′ is parallel to the corresponding equivalence in Theorem II of § 10.

Apparently to get complete equivalence it is necessary either to strengthen condition B′ or add further postulates on the fundamental space. Just what is necessary has not as yet been determined. A sufficient, but not necessary condition is that the space have the property S. Then the property H is valid, which added to B′ gives C′ as in Moore's proof of the Borel Theorem in an $\mathfrak{L}$ space given in § 9. We thus have the following theorem.

THEOREM II′. *If the space* $\mathfrak{V}$ *has the property* S, *then properties* A_1', A_2', B′, *and* C′ *as applied to any set* E *of the space are equivalent.*

We note that since in a space $\mathfrak{V}$ with the property S, the set $\bar{E}=E+E'$ is closed, perfect self-compactness or condition B′ is equivalent to "If $\mathfrak{F}$ is any infinite monotonic family of closed sets such that each set $F\cdot E$ contains an element, then the FE have a common element", the relationship of which to condition B of Theorem II of § 10 is obvious.

Kuratowski and Sierpinski in a space $\mathfrak{L}$ define a $\mathfrak{V}$ space by the condition that any set V to which x is interior is

* Cf. Hausdorff, *Mengenlehre*, 1914, p. 132.

a vicinity of x. Then their statement of the Borel Theorem is as follows:

In an $\mathfrak{L}$ *space having the property* S, *a necessary and sufficient condition that every self-compact set* E *have the Borel any-to-finite property is that every set of the space which is compact and whose derived set is compact have a complete interior limiting element.*

The relationship to Theorem II is apparent.

Finally it is obvious that Theorem II can be generalized as in § 10, giving a theorem corresponding to Theorem III, which contains the Lindelöf Theorem* as a special case, and might be labelled "The Borel-any-to-less-than-power μ Theorem."

In closing we cannot refrain from calling attention to a justification of the consideration of general spaces, in gathering under the same roof such apparently diverse results as the Borel and Lindelöf Theorems, and producing a result of greater scope.

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* For an $\mathfrak{L}$ space, first given by Kuratowski and Sierpinski, loc. cit., pp. 176–8.