

ON THE SERIAL RELATION IN
BOOLEAN ALGEBRAS*

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The elements x, y, z, \dots of a class K are said to form a *series* with respect to a dyadic relation S , if they satisfy the following postulates:†

$P_1.$ If $x \neq y$, then either xSy or ySx .

$P_2.$ If xSy , then $x \neq y$.

$P_3.$ If xSy and ySz , then xSz .

It is my object to determine all serial relations in Boolean algebras given by universal propositions expressible in the fundamental Boolean operations of addition, multiplication, and negation.

All the desired serial relations S must be of the form

$$(1) \quad axy + bxy' + cx'y + dx'y' = 0,$$

where x' is the negative of x . Our problem then reduces itself to finding the conditions imposed on the coefficients of (1) by P_1 - P_3 . We proceed to determine these conditions.

If in (1) $x=0$ and $y=1$, then $c=0$; if $x=1$ and $y=0$, then $b=0$. Hence the condition imposed on (1) by P_1 is

$$(2) \quad b = 0 \quad \text{or} \quad c = 0.$$

The condition imposed on (1) by P_2 is that the equation $axx + bxx' + cx'x + dx'x' = 0$ have no solution. This condition is

$$(3) \quad ad \neq 0.$$

The condition that (1) satisfies P_3 non-vacuously is (this BULLETIN, vol. 30, p. 127)

$$(4) \quad a + d < b + c, \quad ad = 0,$$

which contradicts (3). Hence, there are no non-vacuous serial relations (1) in any Boolean algebra.

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† See E. V. Huntington, *The Continuum*, 2d ed., p. 10.

But there may possibly be *vacuous* serial relations (1). To determine these, let us first find the condition that there be no x, y, z such that both xSy and ySz hold, i. e., the condition that there be no x, y, z such that simultaneously

$$(i) \quad axy + bxy' + cx'y + dx'y' = 0,$$

$$(ii) \quad ayz + byz' + cy'z + dy'z' = 0.$$

Since (i) and (ii) together are equivalent to the single equation

$$(iii) \quad axyz + (b+c)xy'z + (b+c)x'y'z + (c+d)x'y'z \\ + (a+b)xyz' + (b+d)xy'z' + (b+c)x'y'z' + dx'y'z' = 0,$$

our condition is found to be

$$(5) \quad ad(b+c) \neq 0.$$

Now a in (1) must be either 0 or 1, for otherwise neither $1Sa$ nor $aS1$, contrary to P_1 . Similarly b, c, d must be either 0 or 1. Hence, taking account of (2) and (5), we find that all vacuous serial relations (1) must be among the relations

$$(6) \quad xy + xy' + x'y' = 0, \quad xy + x'y + x'y' = 0.$$

Since for $e \neq 0, 1$ neither equation of (6) is satisfied by $x = 1$ and $y = e$, or by $x = e$ and $y = 1$, we see that *there are no vacuous serial relations (1) in a Boolean algebra consisting of more than two elements.*

However, equations (6) are serial relations in a *two-element* Boolean algebra, since they produce respectively the relation-tables (this BULLETIN vol. 30, p. 27):

$$(7) \quad \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & - & + \\ 1 & - & - \end{array}, \quad \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & - & - \\ 1 & + & - \end{array}$$

We may sum up our results as follows: *Relations (6) are the only serial relations of form (1) in Boolean algebras; both of these relations are vacuous and both are confined to the Boolean algebra of order two.*

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