NOTE ON HOROSPHERES*

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In a paper† before the Mathematical Association of America (Sept. 1923) and in another before the International Mathematical Congress at Toronto (August, 1924) I have developed a treatment of a part of non-euclidean geometry which has certain advantages, so it seems to me, over the projective methods of Klein. This method rests on the metric of Riemann and on the introduction of certain variables in terms of which the equations of the straight line and plane are linear.

In the first paper mentioned I showed how easy it is to arrive at Clifford's parallels in elliptic space. These lie on surfaces called Clifford surfaces; their curvature is zero and hence their geometry for restricted regions is euclidean.

In hyperbolic space there are also surfaces of zero curvature, the horospheres of Lobatschevsky and Bolyai. I now wish to show how they may be obtained by the preceding method.

Let x, y, z be ordinary rectangular coördinates. Let R be a positive constant. We set

$$r^2 = x^2 + y^2 + z^2, \quad \lambda = 4R^2 - r^2, \quad \mu = 4R^2 + r^2,$$
$$d\sigma^2 = dx^2 + dy^2 + dz^2. \tag{1}$$

The metric of H-space‡ as defined by Riemann is given by the equation

$$ds = \frac{4R^2 d\sigma}{\lambda} \cdot \tag{2}$$

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† AMERICAN MATHEMATICAL MONTHLY, vol. 30, p. 425, and vol. 31, p. 26. The PROCEEDINGS OF THE TORONTO CONGRESS have not yet appeared.

‡ For H-, read hyperbolic; for e-, read euclidean.

Here R plays the role of the *space constant* and $-1/R^2$ the curvature of this space. Since $ds<0$ for e-points without the e-sphere $\lambda=0$, we restrict ourselves in H-geometry to points within it. The distance from any point within λ to a point on it is infinite. It is the boundary of H-space. Straight lines or H-straights are defined by

$$\delta\int ds = 0.$$

They turn out to be e-circles cutting the λ-sphere orthogonally.

In many problems it is convenient to introduce the variables

$$(3) \qquad z_1 = \frac{4R^2x}{\lambda}, \quad z_2 = \frac{4R^2y}{\lambda}, \quad z_3 = \frac{4R^2z}{\lambda}, \quad z_4 = R\mu/\lambda.$$

In problems of the plane we may set $z=0$ in r, λ, μ in (1) and use

$$(4) \qquad z_1 = \frac{4R^2x}{\lambda}, \quad z_2 = \frac{4R^2y}{\lambda}, \quad z_3 = R\mu/\lambda.$$

The λ sphere is then replaced by the λ-circle, $x^2+y^2-4R^2=0$. With this introduction, let us consider H-circles, in the xy plane. Their equation is in z-coordinates

$$(5) \qquad a_1z_1 + a_2z_2 - a_3z_3 = k, \text{ a constant,}$$

the center having the z-coordinates a_1, a_2, a_3. In xy coordinates (5) becomes

$$(6) \qquad x^2 + y^2 + \frac{4R^2}{k - a_3R}(a_1x + a_2y) - \frac{4R^2(k + a_3R)}{k - a_3R} = 0,$$

which is an e-circle.

We ask what happens when the center a converges to a point on the λ-circle. To answer this question we consider the family of e-circles

$$(7) \qquad C_\alpha = x^2 + y^2 - 2\alpha x - 4R(R - \alpha) = 0.$$

whose centers lie on the x axis and which touch the λ-circle at the point A; also the e-circles

$$L_\beta = x^2 + y^2 - 4Rx - 2\beta y + 4R^2 = 0. \tag{8}$$

These cut the λ-circle orthogonally and are therefore H-straights. They also pass through the point A and cut the C_α orthogonally.

We show that the C_α may be regarded as H-circles whose center is A and whose radii are the L_β. In fact let L_β cut C_α and C_{α_1} in P, P_1; we show the H-length of the segment $P\,P_1$ on the radius L_β is independent of β.

For if L_β cuts C_α in the point P whose coordinates are x, y we find that

$$\xi = x - 2R = \frac{2\beta^2 c}{\gamma^2}, \quad y = \frac{2\beta c^2}{\gamma^2}, \tag{9}$$

where

$$c = \alpha - 2R, \quad \gamma^2 = \beta^2 + c^2. \tag{10}$$

If L_β cuts the adjacent circle $C_{\alpha+d\alpha}$ in P', we find the length of the arc $P\,P'$ is

$$ds = \frac{2R^2 d\alpha}{\alpha(2R-\alpha)};$$

hence

$$\overline{PP_1} = \int ds, \tag{11}$$

which is independent of the particular radius L_β.

We may also show that the H-circles (5) or (6) converge to the circles C_α. In fact take a on the x axis for simplicity; then $a_2 = 0$. We may now let a_1, a_3, $k \to \infty$ such that

$$\lim_{k\to\infty} \frac{a_1}{k} = \lim_{k\to\infty} \frac{a_3}{k} = \frac{1}{R} \cdot \frac{\alpha}{\alpha - 2R}. \tag{12}$$

In this case (6) goes over into a C_α circle. These C_α circles are horocircles. For simplicity we took their H-center A on the x axis; obviously A may be any point on the λ-circle. Any e-circle lying within $\lambda=0$ and tangent to it, is a *horocircle* in H-geometry.

To get a *horosphere* we merely revolve one of the horocircles about the diameter through A; i.e., an e-sphere lying within the λ-sphere or $x^2+y^2+z^2-4R^2=0$, and tangent to it is a horosphere. We wish to show that these surfaces have 0-curvature or that the metric on them has the form

$$ds^2 = \frac{4R^4c^2}{\alpha}(du^2 + u^2d\varphi^2). \tag{13}$$

To this end let us revolve a C_α about the x axis getting a horosphere S_α. The intersection P of C_α and some L_β, describes an e-circle which we may regard as a parallel of latitude β whose pole is A and we may use this as one coordinate. As second coordinate on S_α we may use the meridian circles φ cut out of S_α by e-planes through the x-axis and making the angle φ with the xz plane. If now we set $\rho^2=y^2+z^2$, we have, using (9), (10),

$$\rho = \frac{2\beta c^2}{\gamma^2}, \quad y = \rho\sin\varphi, \quad z = \rho\cos\varphi.$$

Now ds is given by (2). We find

$$\beta d\beta = \gamma d\gamma, \qquad dx = d\xi, \qquad dy^2 + dz^2 = dp^2 + \rho^2 d\varphi^2,$$

$$d\rho = \frac{2c^2}{\gamma^4}(c^2 - \beta^2)d\beta,\ d\xi = \frac{4\beta c^3}{\gamma^4}d\beta,\ d\sigma^2 = \frac{4c^4}{\gamma^4}(d\beta^2 + \beta^2 d\varphi^2).$$

Hence

$$ds^2 = \frac{4R^4c^2}{\alpha^2}\left(\frac{d\beta^2}{\beta^4} + \frac{d\varphi^2}{\beta^2}\right).$$

Set now $\mu=1/\beta$; then ds^2 reduces to (13); q.e.d.

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