SOME THEOREMS CONCERNING MEASURABLE FUNCTIONS*

BY L. M. GRAVES†

Theorems on the measurability of functions of measurable functions, e. g., in the form $F(x) = f[x, g(x)]$, have been given by Carathéodory and other writers.‡ Our Theorem I is an easy generalization of the one given by Carathéodory on page 665, with a slightly different method of proof. Here the function $f(x, y)$ is supposed to be defined for all values of y. Our Theorem II merely applies Theorem I to certain cases when the function $f(x, y)$ is *not* defined for all values of y. In these theorems the variables x and y may be multipartite. Theorems I and II are still valid if, throughout, *measurable* is replaced by *Borel measurable*.

In Theorem III, we consider a summable function $f(x, y)$ of two variables, and show by means of Theorem I that the function of x alone

$$\int_a^x f(x, y) dy$$

is also summable, under a suitable convention.

Notations. In Theorems I and II we use the following abbreviated notations: The point $(x_1, \cdots, x_k)$ in k-dimensional space, we denote simply by x. The x-space as a whole is denoted by the German $\mathfrak{X}$. We do similarly for the m-dimensional space $\mathfrak{Y}$. When we have to speak of the $(k+m)$-dimensional space $(\mathfrak{X}, \mathfrak{Y})$, we may denote

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‡ See Carathéodory, *Vorlesungen über reelle Funktionen*, pp. 376, 377, 665; Hans Hahn, *Theorie der reellen Funktionen*, p. 556.
Hobson, *Theory of Functions of a Real Variable*, 2d ed., vol. 1, p. 518.

it by $\mathfrak{W}$. Corresponding to a set $\mathfrak{W}^{(0)}$ of points of the space $\mathfrak{W}$ and a point y of the space $\mathfrak{Y}$, we denote by $\mathfrak{X}^{(y)}$ the set of all points x such that (x, y) is in $\mathfrak{W}^{(0)}$. The sets $\mathfrak{Y}^{(x)}$ are defined similarly. A set of m functions $g_1(x), \cdots, g_m(x)$, each single-real-valued on a set $\mathfrak{X}^{(0)}$ of the space $\mathfrak{X}$, will be denoted simply by $g(x)$, and called a function *on* $\mathfrak{X}^{(0)}$ *to* $\mathfrak{Y}$. This function is said to be measurable on $\mathfrak{X}^{(0)}$ if each component is measurable. We denote by $[y]_a$ the closed neighborhood of the point y consisting of all those points $\bar{y}$ distant from y by not more than a.

THEOREM I. *Let* $\mathfrak{X}^{(0)}$ *be a measurable set, and let* $f(x, y)$ *be a single-real-valued function on* $\mathfrak{X}^{(0)}\mathfrak{Y}$ *with the properties* (1) f *is measurable on* $\mathfrak{X}^{(0)}$ *for each* y, *and* (2) f *is continuous in each argument* y_j, *either on the right or on the left, when the other variables are fixed. Then if* $g(x)$ *on* $\mathfrak{X}^{(0)}$ *to* $\mathfrak{Y}$ *is measurable on* $\mathfrak{X}^{(0)}$, *so is the function* $f(x, g(x))$.

We take first the case $m=1$, and assume (to fix the ideas) that f is continuous on the left in y. We construct a sequence $\{\pi_n\}$ of partitions of the y-axis, for example by taking the division points in π_n to be

$$l_{ni} = \frac{i}{n}, \qquad (i = -\infty, \cdots, +\infty).$$

Then the set $\mathfrak{X}^{(ni)}$ of points of the measurable set $\mathfrak{X}^{(0)}$ for which $l_{ni} \leqq g(x) < l_{n,\, i+1}$, is measurable, and we have

$$\sum_i \mathfrak{X}^{(ni)} = \mathfrak{X}^{(0)}$$

for every n. We construct a sequence $\{g_n(x)\}$ of functions measurable on $\mathfrak{X}^{(0)}$ and approaching $g(x)$ from the left by setting $g_n(x) = l_{ni}$ on the set $\mathfrak{X}^{(ni)}$. Hence the function $f(x, g_n(x))$, which equals $f(x, l_{ni})$ on the set $\mathfrak{X}^{(ni)}$, is measurable on $\mathfrak{X}^{(ni)}$, and therefore measurable on $\mathfrak{X}^{(0)}$. Since f is continuous on the left in y, we have $\lim f(x, g_n(x)) = f(x, g(x))$, and the last named function is also measurable on $\mathfrak{X}^{(0)}$.

We complete the proof by induction. By the theorem for m, $f(x, g(x), y_{m+1})$ is measurable on $\mathfrak{X}^{(0)}$ and continuous (right or left) in y_{m+1}. Hence, by the proof just given, $f(x, g(x), g_{m+1}(x))$ is measurable.

THEOREM II. *Let the set* $\mathfrak{W}^{(0)}$ *and the function* $f(x, y)$ *single-real-valued on* $\mathfrak{W}^{(0)}$ *be such that* (1) *for each* y, f *is measurable in* x *on every measurable set contained in* $\mathfrak{X}^{(y)}$, *and* (2) *for each* x, f *is continuous on* y *in* $\mathfrak{Y}^{(x)}$. *Let* $\mathfrak{X}^{(0)}$ *be a measurable set, and let* $g(x)$ *be a function on* $\mathfrak{X}^{(0)}$ *to* $\mathfrak{Y}$, *which is measurable on* $\mathfrak{X}^{(0)}$, *and such that for a fixed positive number* a, *the neighborhood* $[g(x)]_a$ *is in* $\mathfrak{Y}^{(x)}$ *for every* x. *Then the function* $f(x, g(x))$ *is measurable on* $\mathfrak{X}^{(0)}$.

Divide the space $\mathfrak{Y}$ into a denumerable infinity of "cubes" $\mathfrak{Y}^{(j)}$, with edges parallel to the axes of the space, and maximum diameter less than or equal to the number a. Let $\mathfrak{X}^{(j)}$ be the subset of $\mathfrak{X}^{(0)}$ on which $g(x)$ is in the set $\mathfrak{Y}^{(j)}$. Then each $\mathfrak{X}^{(j)}$ is measurable, being a product of measurable sets, and $\sum \mathfrak{X}^{(j)} = \mathfrak{X}^{(0)}$. We consider hereafter only those values of j for which $\mathfrak{X}^{(j)}$ is not empty. Let $y^{(j)}$ be the center of the "cube" $\mathfrak{Y}^{(j)}$. Then for every x in $\mathfrak{X}^{(j)}$, the point $g(x)$ is contained in the closed neighborhood $[y^{(j)}]_b$ (where $2b = a$), and the neighborhood $[y^{(j)}]_b$ in turn is contained in the neighborhood $[g(x)]_a$ and hence in the set $\mathfrak{Y}^{(x)}$. We can now define a function $F(x, y)$ on $\mathfrak{X}^{(j)}\mathfrak{Y}$, equal to $f(x, y)$ on $\mathfrak{X}^{(j)}[y^{(j)}]_b$, measurable on $\mathfrak{X}^{(j)}$ for every y, and continuous on $\mathfrak{Y}$ for every x. E. g., for points y not in $[y^{(j)}]_b$, set $F(x, y) = f(x, y^{(j)} + c(y - y^{(j)}))$, where $b = c \times$ distance from y to $y^{(j)}$. Then by Theorem I, $F(x, g(x)) = f(x, g(x))$ is measurable on the set $\mathfrak{X}^{(j)}$. Hence $f(x, g(x))$ is measurable on $\mathfrak{X}^{(0)}$.

In the proof of Theorem III, we shall need the following preliminary theorem. (We now drop the abbreviated notation of the preceding paragraphs.)

Suppose the single-real-valued function $f(x, y)$ *is summable on the rectangle* $a \leqq x \leqq b$, $c \leqq y \leqq d$. *Then there exists a set* $\mathfrak{E}$ *of points of the interval* (a, b) *such that*:

(1) *measure of* $\mathfrak{E} = b - a$;

(2) *the integral*

$$\int_c^y f(x, y)dy = g(x, y)$$

exists for every x in the set $\mathfrak{E}$ and every y in (c, d);

(3) *$g(x, y)$ is measurable in x on $\mathfrak{E}$, for every y;*

(4) *$|g(x, y)| \leqq M(x)$ for every y, where $M(x)$ is summable on $\mathfrak{E}$.*

When we take $y=d$, the statements of this theorem are at least implicitly contained in every treatment of the reduction of a double integral of a summable function to two successive simple integrals.* We obtain the theorem stated for a value $y=y_0<d$ by replacing $f(x, y)$ by a function $f_0(x, y)$, equal to f for $c \leqq y \leqq y_0$, and zero for $y_0<y\leqq d$. It is readily seen that the set $\mathfrak{E}$ effective for $y=d$ is effective for all values of y. To obtain the fourth conclusion, we have

$$| g(x,y) | \leqq \int_c^y | f(x,y) | \, dy \leqq \int_c^y | f(x,y) | \, dy.$$

THEOREM III. *Suppose the function $f(x, y)$ is summable on the square $a \leqq x \leqq b$, $a \leqq y \leqq b$. Then there exists a set $\mathfrak{E}$ of points of (a, b), whose measure is $(b-a)$, such that the function*

$$\int_a^x f(x,y)dy$$

is summable on $\mathfrak{E}$.

By our preliminary theorem, the function

$$g(x,y) = \int_a^y f(x,y)dy$$

is measurable in x on a set $\mathfrak{E}$ with the specified properties, and satisfies the condition $|g(x, y)| \leqq M(x)$, where $M(x)$ is summable on $\mathfrak{E}$. It is also continuous in y on (a, b). Hence we can extend the range of definition of the function $g(x, y)$ so that the conditions of Theorem I will be satisfied. This, with the inequality $|g(x, x)| \leqq M(x)$, shows that $g(x, x)$ is summable on $\mathfrak{E}$.

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* See de la Vallée Poussin, *Intégrales de Lebesgue*, pp. 50–53; or BULLETIN DE L'ACADÉMIE DE BELGIQUE, Sciences, 1910, p. 768.

Various modifications of Theorem III may easily be secured. For example, in case we make the additional assumptions that the function $f(x, y)$ is bounded and is measurable in y for each x, then the set $\mathfrak{E}$ may be replaced by the interval (a, b). These additional assumptions are fulfilled in particular if f is bounded and Borel measurable on the square where it is defined. In this case the function $g(x, x)$ is Borel measurable on (a, b). As another modification we may substitute for the square $a \leqq x \leqq b$, $a \leqq y \leqq b$, a bounded measurable set $\mathfrak{E}_0\mathfrak{E}_0$, consisting of those points of the plane having x and y each in a linear measurable set $\mathfrak{E}_0$. Then the integral is understood to be taken over those points of the interval (a, x) contained in $\mathfrak{E}_0$.

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A GENERAL THEORY OF REPRESENTATION OF FINITE OPERATIONS AND RELATIONS*

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Let a mod n denote the least positive residue modulo n of an integer a, i. e., the least positive integer obtained from a by rejecting multiples of n. Consider the polynomials modulo a prime p

(1) $$a_0 + a_1 x + \cdots + a_{p-1}x^{p-1}, \text{ mod } p,$$

(2) $$f_0(x) + f_1(x)y + \cdots + f_{p-1}(x)y^{p-1}, \text{ mod } p,$$

where in (1) a_i are least positive p-residues and x ranges over the complete system of least positive p-residues, and where (2) is a polynomial modulo p in y whose coefficients $f_i(x)$ are modular polynomials in x of form (1). In a previous paper† I developed a theory of representation of abstract

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† PROCEEDINGS OF THE INTERNATIONAL MATHEMATICAL CONGRESS, TORONTO, 1924.