

SOME THEOREMS CONCERNING  
MEASURABLE FUNCTIONS\*

BY L. M. GRAVES†

Theorems on the measurability of functions of measurable functions, e. g., in the form  $F(x) = f[x, g(x)]$ , have been given by Carathéodory and other writers.‡ Our Theorem I is an easy generalization of the one given by Carathéodory on page 665, with a slightly different method of proof. Here the function  $f(x, y)$  is supposed to be defined for all values of  $y$ . Our Theorem II merely applies Theorem I to certain cases when the function  $f(x, y)$  is *not* defined for all values of  $y$ . In these theorems the variables  $x$  and  $y$  may be multipartite. Theorems I and II are still valid if, throughout, *measurable* is replaced by *Borel measurable*.

In Theorem III, we consider a summable function  $f(x, y)$  of two variables, and show by means of Theorem I that the function of  $x$  alone

$$\int_a^x f(x, y) dy$$

is also summable, under a suitable convention.

*Notations.* In Theorems I and II we use the following abbreviated notations: The point  $(x_1, \dots, x_k)$  in  $k$ -dimensional space, we denote simply by  $x$ . The  $x$ -space as a whole is denoted by the German  $\mathfrak{X}$ . We do similarly for the  $m$ -dimensional space  $\mathfrak{Y}$ . When we have to speak of the  $(k+m)$ -dimensional space  $(\mathfrak{X}, \mathfrak{Y})$ , we may denote

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‡ See Carathéodory, *Vorlesungen über reelle Funktionen*, pp. 376, 377, 665; Hans Hahn, *Theorie der reellen Funktionen*, p. 556. Hobson, *Theory of Functions of a Real Variable*, 2d ed., vol. 1, p. 518.

it by  $\mathfrak{B}$ . Corresponding to a set  $\mathfrak{B}^{(0)}$  of points of the space  $\mathfrak{B}$  and a point  $y$  of the space  $\mathfrak{Y}$ , we denote by  $\mathfrak{X}^{(y)}$  the set of all points  $x$  such that  $(x, y)$  is in  $\mathfrak{B}^{(0)}$ . The sets  $\mathfrak{Y}^{(x)}$  are defined similarly. A set of  $m$  functions  $g_1(x), \dots, g_m(x)$ , each single-real-valued on a set  $\mathfrak{X}^{(0)}$  of the space  $\mathfrak{X}$ , will be denoted simply by  $g(x)$ , and called a function on  $\mathfrak{X}^{(0)}$  to  $\mathfrak{Y}$ . This function is said to be measurable on  $\mathfrak{X}^{(0)}$  if each component is measurable. We denote by  $[y]_a$  the closed neighborhood of the point  $y$  consisting of all those points  $\bar{y}$  distant from  $y$  by not more than  $a$ .

**THEOREM I.** *Let  $\mathfrak{X}^{(0)}$  be a measurable set, and let  $f(x, y)$  be a single-real-valued function on  $\mathfrak{X}^{(0)}\mathfrak{Y}$  with the properties (1)  $f$  is measurable on  $\mathfrak{X}^{(0)}$  for each  $y$ , and (2)  $f$  is continuous in each argument  $y_i$ , either on the right or on the left, when the other variables are fixed. Then if  $g(x)$  on  $\mathfrak{X}^{(0)}$  to  $\mathfrak{Y}$  is measurable on  $\mathfrak{X}^{(0)}$ , so is the function  $f(x, g(x))$ .*

We take first the case  $m=1$ , and assume (to fix the ideas) that  $f$  is continuous on the left in  $y$ . We construct a sequence  $\{\pi_n\}$  of partitions of the  $y$ -axis, for example by taking the division points in  $\pi_n$  to be

$$l_{ni} = \frac{i}{n}, \quad (i = -\infty, \dots, +\infty).$$

Then the set  $\mathfrak{X}^{(ni)}$  of points of the measurable set  $\mathfrak{X}^{(0)}$  for which  $l_{ni} \leq g(x) < l_{n,i+1}$ , is measurable, and we have

$$\sum_i \mathfrak{X}^{(ni)} = \mathfrak{X}^{(0)}$$

for every  $n$ . We construct a sequence  $\{g_n(x)\}$  of functions measurable on  $\mathfrak{X}^{(0)}$  and approaching  $g(x)$  from the left by setting  $g_n(x) = l_{ni}$  on the set  $\mathfrak{X}^{(ni)}$ . Hence the function  $f(x, g_n(x))$ , which equals  $f(x, l_{ni})$  on the set  $\mathfrak{X}^{(ni)}$ , is measurable on  $\mathfrak{X}^{(ni)}$ , and therefore measurable on  $\mathfrak{X}^{(0)}$ . Since  $f$  is continuous on the left in  $y$ , we have  $\lim f(x, g_n(x)) = f(x, g(x))$ , and the last named function is also measurable on  $\mathfrak{X}^{(0)}$ .

We complete the proof by induction. By the theorem for  $m$ ,  $f(x, g(x), y_{m+1})$  is measurable on  $\mathfrak{X}^{(0)}$  and continuous

(right or left) in  $y_{m+1}$ . Hence, by the proof just given,  $f(x, g(x), g_{m+1}(x))$  is measurable.

**THEOREM II.** *Let the set  $\mathfrak{B}^{(0)}$  and the function  $f(x, y)$  single-real-valued on  $\mathfrak{B}^{(0)}$  be such that (1) for each  $y$ ,  $f$  is measurable in  $x$  on every measurable set contained in  $\mathfrak{X}^{(y)}$ , and (2) for each  $x$ ,  $f$  is continuous on  $y$  in  $\mathfrak{Y}^{(x)}$ . Let  $\mathfrak{X}^{(0)}$  be a measurable set, and let  $g(x)$  be a function on  $\mathfrak{X}^{(0)}$  to  $\mathfrak{Y}$ , which is measurable on  $\mathfrak{X}^{(0)}$ , and such that for a fixed positive number  $a$ , the neighborhood  $[g(x)]_a$  is in  $\mathfrak{Y}^{(x)}$  for every  $x$ . Then the function  $f(x, g(x))$  is measurable on  $\mathfrak{X}^{(0)}$ .*

Divide the space  $\mathfrak{Y}$  into a denumerable infinity of "cubes"  $\mathfrak{Y}^{(j)}$ , with edges parallel to the axes of the space, and maximum diameter less than or equal to the number  $a$ . Let  $\mathfrak{X}^{(j)}$  be the subset of  $\mathfrak{X}^{(0)}$  on which  $g(x)$  is in the set  $\mathfrak{Y}^{(j)}$ . Then each  $\mathfrak{X}^{(j)}$  is measurable, being a product of measurable sets, and  $\sum \mathfrak{X}^{(j)} = \mathfrak{X}^{(0)}$ . We consider hereafter only those values of  $j$  for which  $\mathfrak{X}^{(j)}$  is not empty. Let  $y^{(j)}$  be the center of the "cube"  $\mathfrak{Y}^{(j)}$ . Then for every  $x$  in  $\mathfrak{X}^{(j)}$ , the point  $g(x)$  is contained in the closed neighborhood  $[y^{(j)}]_b$  (where  $2b = a$ ), and the neighborhood  $[y^{(j)}]_b$  in turn is contained in the neighborhood  $[g(x)]_a$  and hence in the set  $\mathfrak{Y}^{(x)}$ . We can now define a function  $F(x, y)$  on  $\mathfrak{X}^{(j)}\mathfrak{Y}$ , equal to  $f(x, y)$  on  $\mathfrak{X}^{(j)}[y^{(j)}]_b$ , measurable on  $\mathfrak{X}^{(j)}$  for every  $y$ , and continuous on  $\mathfrak{Y}$  for every  $x$ . E. g., for points  $y$  not in  $[y^{(j)}]_b$ , set  $F(x, y) = f(x, y^{(j)} + c(y - y^{(j)}))$ , where  $b = c \times$  distance from  $y$  to  $y^{(j)}$ . Then by Theorem I,  $F(x, g(x)) = f(x, g(x))$  is measurable on the set  $\mathfrak{X}^{(j)}$ . Hence  $f(x, g(x))$  is measurable on  $\mathfrak{X}^{(0)}$ .

In the proof of Theorem III, we shall need the following preliminary theorem. (We now drop the abbreviated notation of the preceding paragraphs.)

*Suppose the single-real-valued function  $f(x, y)$  is summable on the rectangle  $a \leq x \leq b$ ,  $c \leq y \leq d$ . Then there exists a set  $\mathfrak{E}$  of points of the interval  $(a, b)$  such that:*

- (1) *measure of  $\mathfrak{E} = b - a$ ;*
- (2) *the integral*

$$\int_c^y f(x, y) dy = g(x, y)$$

exists for every  $x$  in the set  $\mathfrak{E}$  and every  $y$  in  $(c, d)$ ;

(3)  $g(x, y)$  is measurable in  $x$  on  $\mathfrak{E}$ , for every  $y$ ;

(4)  $|g(x, y)| \leq M(x)$  for every  $y$ , where  $M(x)$  is summable on  $\mathfrak{E}$ .

When we take  $y=d$ , the statements of this theorem are at least implicitly contained in every treatment of the reduction of a double integral of a summable function to two successive simple integrals.\* We obtain the theorem stated for a value  $y=y_0 < d$  by replacing  $f(x, y)$  by a function  $f_0(x, y)$ , equal to  $f$  for  $c \leq y \leq y_0$ , and zero for  $y_0 < y \leq d$ . It is readily seen that the set  $\mathfrak{E}$  effective for  $y=d$  is effective for all values of  $y$ . To obtain the fourth conclusion, we have

$$|g(x, y)| \leq \int_c^y |f(x, y)| dy \leq \int_c^d |f(x, y)| dy.$$

**THEOREM III.** Suppose the function  $f(x, y)$  is summable on the square  $a \leq x \leq b$ ,  $a \leq y \leq b$ . Then there exists a set  $\mathfrak{E}$  of points of  $(a, b)$ , whose measure is  $(b-a)$ , such that the function

$$\int_a^x f(x, y) dy$$

is summable on  $\mathfrak{E}$ .

By our preliminary theorem, the function

$$g(x, y) = \int_a^y f(x, y) dy$$

is measurable in  $x$  on a set  $\mathfrak{E}$  with the specified properties, and satisfies the condition  $|g(x, y)| \leq M(x)$ , where  $M(x)$  is summable on  $\mathfrak{E}$ . It is also continuous in  $y$  on  $(a, b)$ . Hence we can extend the range of definition of the function  $g(x, y)$  so that the conditions of Theorem I will be satisfied. This, with the inequality  $|g(x, x)| \leq M(x)$ , shows that  $g(x, x)$  is summable on  $\mathfrak{E}$ .

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\* See de la Vallée Poussin, *Intégrales de Lebesgue*, pp. 50-53; or BULLETIN DE L'ACADÉMIE DE BELGIQUE, Sciences, 1910, p. 768.

Various modifications of Theorem III may easily be secured. For example, in case we make the additional assumptions that the function  $f(x, y)$  is bounded and is measurable in  $y$  for each  $x$ , then the set  $\mathfrak{E}$  may be replaced by the interval  $(a, b)$ . These additional assumptions are fulfilled in particular if  $f$  is bounded and Borel measurable on the square where it is defined. In this case the function  $g(x, x)$  is Borel measurable on  $(a, b)$ . As another modification we may substitute for the square  $a \leq x \leq b, a \leq y \leq b$ , a bounded measurable set  $\mathfrak{E}_0 \mathfrak{E}_0$ , consisting of those points of the plane having  $x$  and  $y$  each in a linear measurable set  $\mathfrak{E}_0$ . Then the integral is understood to be taken over those points of the interval  $(a, x)$  contained in  $\mathfrak{E}_0$ .

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## A GENERAL THEORY OF REPRESENTATION OF FINITE OPERATIONS AND RELATIONS\*

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Let  $a \bmod n$  denote the least positive residue modulo  $n$  of an integer  $a$ , i. e., the least positive integer obtained from  $a$  by rejecting multiples of  $n$ . Consider the polynomials modulo a prime  $p$

$$(1) \quad a_0 + a_1 x + \cdots + a_{p-1} x^{p-1}, \bmod p,$$

$$(2) \quad f_0(x) + f_1(x)y + \cdots + f_{p-1}(x)y^{p-1}, \bmod p,$$

where in (1)  $a_i$  are least positive  $p$ -residues and  $x$  ranges over the complete system of least positive  $p$ -residues, and where (2) is a polynomial modulo  $p$  in  $y$  whose coefficients  $f_i(x)$  are modular polynomials in  $x$  of form (1). In a previous paper† I developed a theory of representation of abstract

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