

## PRINGSHEIM ON COMPLEX VARIABLES

*Vorlesungen ueber Zahlen- und Funktionenlehre.* Vol. II, pt. 1: *Grundlagen der Theorie der analytischen Funktionen einer komplexen Veraenderlichen.*  
By Alfred Pringsheim. Leipzig, B. G. Teubner, 1925. xv+626 pp.\*

In laying the foundations for analytic functions of a complex variable it is possible to proceed from three practically equivalent points of view, the Meray-Weierstrass, which bases its developments upon the properties of power series, the Cauchy-Riemann point of view, in which the functions to be considered are limited by the existence of a differential quotient, and what might be called the Cauchy-Morera point of view, in which the properties of the function depend upon the properties of the curvilinear integral. Of these, the first two seem to be the only ones which are practicable, and are usually considered simultaneously, each contributing towards the development of a simple and elegant theory. Only occasionally do we find the one emphasized to the almost complete exclusion of the others.

The volume under discussion has for its avowed purpose the development of the foundations of the theory of analytic functions as far as possible along the Meray-Weierstrass lines, i.e., it is based on power series and their properties. This is probably a consistent point of view, in consideration of the fact that this volume is the second in a series, the first of which was devoted to series and sequences of numbers. The choice and arrangement of material is governed then by the desire to accomplish as much as possible by the use of power series.

A brief survey of the contents of the book may be of value.

The first chapter, of an introductory character, is devoted to the real variable and some of the essential properties of functions of real variables. We find, then, discussions of the one to one correspondence between real numbers and points on a line, limits of functions, and properties of continuous functions. The introduction of two variables leads to a consideration of regions and their boundaries, and this in turn to the Jordan Curve Theorem. The method of attack is via the properties of what Pringsheim calls "Treppenvpolygone," which one might translate "stair polygons," consisting of segments of straight lines parallel to the coordinate axes, no point on the boundary occurring twice. This section is of a rather more advanced character than most of the remainder of the book. There is further a brief treatment of double and iterated limits and corresponding properties of functions. The chapter closes by indicating as the reason for adopting the power series as the form of function to be considered, its property of being determined throughout its region of existence by its values at an infinity of arbitrarily chosen points, and the desirability of using complex numbers, as giving perhaps a method for continuing power

---

\* Volume I was reviewed for this BULLETIN by C. N. Moore, part 1, in vol. 25 (1919), p. 470; parts 2 and 3, in vol. 28 (1923), pp. 63-5.

series in real variables beyond certain points of divergence. From an elementary point of view neither of these reasons seems convincing. The determination of a function by an infinity of values comes rather late in experience, and the fact that the power series is the medium through which this can be accomplished is not exactly self-evident.\* The same is true of the notion of continuation as applied to power series. Of course, as indicating two of the important things to be expected of the theory, these two points are well taken.

The second chapter treats in the first place, the complex number, its geometric representation, addition, subtraction, multiplication, and division and their geometric equivalents. The author postpones the introduction of the polar form of the complex number to a later point, so the discussion of multiplication from the geometric aspect is slightly awkward. Functions of a complex variable are introduced and the desirability of considering a special class of these functions indicated. Special consideration from the point of view of transformation is given in the usual fashion to the elementary functions, the linear integral function, the linear fractional function, and the function  $y = x^2$  and its inverse.

The third chapter is concerned with rational functions of the complex variable. The integral function is considered first and a proof of the fundamental theorem of algebra given. The greater part of this chapter is then of an algebraic character, including the Lagrange formula of interpolation, the expression of symmetric functions of the roots as rational functions of the coefficients of a polynomial, and the expression of a rational fractional function in terms of partial fractions.

The fourth chapter turns to the consideration of properties of power series. We find first a general introduction to the convergence of sequences and series of functions, including due emphasis on uniform convergence and the Weierstrassian maximal convergence. This leads readily to the properties of the function defined by power series in the interior of the circle of convergence. The behavior of the series in approaching a point of convergence of the series on the circle of convergence is also discussed. In order to obtain the Cauchy theorem on the upper limit of the values of the coefficients of a power series, Pringsheim finds it desirable to introduce the notion of mean value of a function on a circle inside its region of definition. This is based on the possibility of determining the  $2^n$ th roots of unity by algebraic processes. If  $c_n$  is the  $2^n$ th root of unity with maximum real part and positive imaginary coefficient, then the mean value of the function  $f$  on the circle of radius  $r$  around the origin is defined as

$$\lim_{n \rightarrow \infty} \frac{1}{2^n} \sum_{k=1}^{2^n-1} f(c_n^k r)$$

---

\* It is an interesting problem (to our knowledge still unsolved) to determine what conditions must be satisfied by the values of a function at a denumerably infinite set of points in a finite region so that they determine an analytic function or a power series, and to determine the form of the function in question.

which limit exists if the function  $f$  is continuous on the circle. This mean value is obviously (as Pringsheim points out in the preface) identical with the Cauchy line integral

$$\frac{1}{2\pi i} \int \frac{f(z)dz}{z}$$

along the circle of radius  $r$  around the origin. This mean value enables him not only to obtain the limits for the coefficients but also the coefficients of the power series with positive and negative exponents, in terms of the value of the function along a circle in the region of convergence, including also an expression for the power series itself.

Whether the omission of the line integral and its replacement by the mean value is justifiable is open to question. It is true that in power series considerations, the integral around a circle is paramount. Also the trend in modern mathematics seems to be towards the consideration of mean values rather than integrals only. However, it is not entirely clear how the mean value notion can be effectively extended to curves other than circles. Moreover in the line integral, we possess a powerful tool which produces results which do not seem to be possible by other means.

There follow theorems on the rearrangement of series of power series into power series, viz. the Cauchy, Weierstrass and Vitali theorems, and in the proof of the latter, use is made of the properties of the coefficients obtained through the mean value. A natural sequel is the expansion of  $f(x+h)$  in powers of  $h$ , the definition of the derivatives of the function as the coefficients of  $h^n/n!$ , the properties of derivatives so defined and questions relating to the continuation of power series.

Chapter V is concerned with the definition and properties of monogenic analytic functions. The definition is the usual one, as the totality of power series which can be derived from a given one by analytic continuation. We have the result that such a function can be expanded in the neighborhood of any point interior to the region of existence, i.e. every such point is a regular point. This leads to a consideration of the equivalence of the notion of uniform differentiability (i.e. the existence of the

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

uniformly) in a neighborhood of a point and regularity. We find then also at this stage the Cauchy-Riemann differential equations and a discussion of their role in the determination of the real and imaginary parts of an analytic function.

Pringsheim justifies his choice of power series as a method of attack on the theory of functions of a complex variable in part by its close relationship to arithmetic expressions (i.e. expressions obtainable from  $x$  by the four fundamental operations and limits of sequences). It is the simplest sequel to the rational integral function via the use of the limit process. He holds that the definition of a class of functions by means of a property (in this case the existence of the differential quotient) is not so satisfying.

In this instance, the end justifies the means. On the other hand, one would hardly be tempted to base a theory of continuous functions on arithmetic expressions, to wit, treating them as the functions defined by uniformly convergent sequences of polynomials.

This chapter contains also a proof of the theorem that a function regular in a simply connected region is necessarily single-valued, this being the main use made of the results on Jordan curves of the first chapter. The chapter concludes with the types of possible singularities of analytic functions and a discussion of their characteristics.

The two final chapters of the book are concerned with the elementary transcendental functions and their inverses. The exponential function is derived as the simplest function without singularities and zeros in the finite part of the plane. Its identity with the function

$$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n$$

is then shown. The sine and cosine functions are defined in terms of the exponential function and their properties deduced mainly from their power series expressions. The infinite product forms of these functions are developed as well as the partial fraction expressions of the quotient functions, which in turn lead to a consideration of the Bernoulli and Euler numbers. The chapter contains also a proof of the transcendental character of  $e$  and  $\pi$ . After a brief general consideration of the inversion of power series at the beginning of the last chapter, the theory thus derived is applied to the logarithmic function and the inverse trigonometric functions. A complete discussion of the nature of the general exponent  $b^a$  where  $b$  and  $a$  are complex numbers, occurs here also.

An appendix containing references for further study, and comments additional to those contained in footnotes concludes the volume.

As an introduction to the study of power series and their possibilities, this volume is admirable. It gives a consistent and excellent treatment of what can be done in the theory of functions of a complex variable from this point of view, though it does occasionally sacrifice in elegance and completeness to attain this end. So while it is an interesting exposition, it does seem unsatisfactory as an introduction to the subject of analytic functions in its omission of the line integral and its elegant and important consequences. Perhaps these are to be taken up in the second part of this volume.

The presentation of the subject matter is excellent and clear, and sins rather on the side of too much rather than too little explanation. As a whole the book seems rather diffuse, there are a number of repetitions, due to the method pursued, and it leaves rather the impression of a compendium on the elementary phases of the subject than an introduction, especially in its inclusion of many topics not usually considered in books on this subject.

T. H. HILDEBRANDT