# ON IRREDUNDANT SETS OF POSTULATES

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In a recent paper* Dr. H. M. Gehman has made the point that each of the irredundant sets of postulates proposed by the author in a previous paper† can be obtained from another set, which is not only not irredundant but not even independent, by the mechanical method given by the author and restated by Gehman in his paper.

This mechanical process converts any independent set of postulates into an irredundant set, and, as pointed out by Gehman, it has the same effect on certain non-independent sets. But, applied to an average set of postulates, this process yields postulates which are more or less complicated mixtures of irrelevant ideas. If an irredundant set of postulates is to be of any interest, the postulates of the set ought not to be such mixtures of irrelevant ideas; but the fact that the set can be obtained by the mechanical process from some other set is not an objection to it.

Indeed, any irredundant set can be considered as obtained by the mechanical method. For let $A$, $B$, $C$ be irredundant. In view of the irredundance, $B$ is equivalent to *if* $A$ *then* $AB$, where $AB$ means $A$ *and* $B$. This equivalence is strict. It does not depend on the presence of other postulates. Similarly $C$ is equivalent to *if* $A$ *and* $AB$ *then* $ABC$. Accordingly the set $A, B, C$ can be restated in the form $A$, *if* $A$ *then* $AB$, *if* $A$ *and* $AB$ *then* $ABC$. It is then clear that this set can be obtained by the mechanical method from the set $A, AB, ABC$. And the latter set is not even independent.

In this way, given any irredundant set, we can give it the form of a set obtained by the mechanical method if we

*This BULLETIN, vol. 32 (1926), pp. 159–161.

†TRANSACTIONS OF THIS SOCIETY, vol. 27 (1925), pp. 318–328.

introduce suitable redundancies into the statements of the postulates.

Gehman has used exactly this device. Using Gehman s abbreviated terminology, the author's postulates for a finite cycle were:

1. An $S$-set exists.
2. No proper part of $N$ is an $S$-set.

Gehman restates the second postulate as follows:

$B_2$. If an $S$-set exists, $N$ is the only $S$-set.

This is, of course, simply an abbreviation for:

$B_2$. If an $S$-set exists, $N$ is an $S$-set, and no set other than $N$ is an $S$-set.

This is highly redundant. The clause "$N$ is an $S$-set" is superfluous,* and when it is dropped "If an $S$-set exists" becomes equally so.† As a matter of fact, $B_2$ is simply the postulate *if* 1 *then* 1 *and* 2, plus additional redundancies.

Gehman proposes to replace 1 and 2 by the single postulate:

$A_2$. $N$ is the only $S$-set.

But $A_2$ clearly ought to be separated into the two postulates:

$N$ is an $S$-set.

No set other than $N$ is an $S$-set.

If we weaken the first of these to read "An $S$-set exists," we have postulates 1 and 2 again.

The postulates proposed by the author for the system of positive and negative integers were the following:

1. An $S$-set exists.
2. If $N$ is an $S$-set, it is not the only $S$-set.
3. If an $S$-set exists, an $ST$-set exists.
4. No proper part of $N$ is an $ST$-set.

Gehman replaces 4 by:

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*Because if an $S$-set exists, and no set other than $N$ is an $S$-set, then necessarily $N$ is an $S$-set.

†Because if an $S$-set does not exist it follows at once that no set other than $N$ is an $S$-set.

$C_4$. If an $ST$-set exists, $N$ is the only $ST$-set.

This substitution introduces redundancies in exactly the same way as before. Postulate 3 of this set, however, stands without alteration in the form, *if* 1 *then so and so.* This is possibly a reason for combining 1 and 3 into the single postulate, "An $ST$-set exists," a procedure which would not, of course, alter the irredundance of the set of postulates. But the retention of 1 and 3 as separate postulates is defensible on the ground that no irrelevancies among the parts of any postulate are thereby introduced.

Gehman has shown that postulates 1, 3, and 4 can be derived from another set by the mechanical process. There is no reason for the omission of postulate 2, because it is equally true that postulates 1, 2, 3, and 4 can be derived from another set by the mechanical process.

For example, arranged in the order 4, 3, 1, 2, they can be derived by the mechanical method from the following postulates:

4°. No proper part of $N$ is an $ST$-set.

3°. If an $S$-set exists, $N$ is the only $ST$-set.

1°. $N$ is the only $ST$-set.

2°. $N$ is the only $ST$-set but not the only $S$-set.

A similar statement is true of the postulates arranged in any other order. In fact, not only can the set of postulates be derived by the mechanical process from a non-independent set, but it can be so derived from each of 4! different sets no one of which is independent. And the same thing is true of any irredundant set of four postulates.

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