ON THE INTEGRO-DIFFERENTIAL EQUATION OF THE BÔCHER TYPE IN THREE-SPACE

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1. *Introduction.* Bôcher has shown* that if a function $f(x, y)$ is continuous and has continuous first partial derivatives in a region R and satisfies the condition

$$\int_C \frac{\partial f}{\partial n}\, ds = 0$$

for every circle C lying entirely in R, then $f(x, y)$ is harmonic at each interior point of R. Bôcher treats only functions in two variables and by a method which cannot be directly extended to three-space.

It is the purpose of the present note to show, by a simple modification of the second part of Bôcher's argument, that this result may at once be extended to three-space, and also to investigate the nature of the function f if Bôcher's condition of continuity is somewhat weakened. We shall treat explicitly functions in three variables only, but it will easily be seen that with a slight modification the statements of Theorem II are applicable to two-space as well.

THEOREM I. *If a function $f(x, y, z)$ is continuous, and has continuous first partial derivatives in a connected finite region R, and is such that the surface integral $\int_S (\partial f/\partial n) ds$ vanishes when taken over every sphere S lying in R, then at each interior point of R, f is harmonic; that is, it satisfies Laplace's equation*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

at each interior point of R.

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* PROCEEDINGS OF THE AMERICAN ACADEMY, vol. 41, pp. 577–583.

In the preceding integral, as well as in what follows, the derivative $\partial f/\partial n$ is to be taken either toward the interior of S, or toward the exterior of S, throughout the region of integration. Let P be any interior point of R and consider two spheres S_1 and S_2 of radii r_1 and $r_2<r_1$ with centers at P. By hypothesis, we have

$$\int_S \frac{\partial f}{\partial n}\, ds = 0\ ;$$

or, setting $ds=r^2d\omega$, where $d\omega$ is the element of area on the unit sphere with center P,

$$\int_S \frac{\partial f}{\partial n}\, d\omega = 0.$$

It follows that

$$(1)\qquad \int_{r_2}^{r_1} dr \int_S \frac{\partial f}{\partial n}\, d\omega = 0.$$

Because of the continuity of f and its derivatives the order of integrations in the above integral may be inverted and we have

$$(2)\qquad \int_{S_1} f d\omega - \int_{S_2} f d\omega = 0.$$

Let $f(P)$ be the value of f at the point P. Then since f is continuous at P we obtain from (2) by letting r_2 approach zero,

$$f(P) = \frac{1}{4\pi}\int_{S_1} f d\omega = \frac{1}{4\pi r_1^2}\int_{S_1} f ds.$$

We thus see that our function f possesses the so-called *mean-value property*, that is, its value at the center of any sphere is the mean of its values on the surface of the sphere.

Consider now the function F which takes the same values as f on S_1 and which is harmonic interior to S_1. This function exists and can be expressed as a Poisson integral. It is well known that F also possesses the mean-value property and hence so also does the difference $f-F$. But a continuous function having the mean value property in a closed region R must take its greatest and least values on the boundary of R. Since the difference $f-F$ is identically zero on S_1 it follows that it is zero everywhere within S_1 and hence f must be harmonic at P as was to be proved.

It is evident that the original hypothesis that

$$\int \frac{\partial f}{\partial n} ds = 0$$

about every sphere in R is unnecessarily broad. All that is needed in the above proof is that each point P may be surrounded by a region, no matter how small, which is such that the above integral vanishes when taken over every sphere lying entirely within it.

2. *A More General Theorem.* We shall now weaken the original condition of continuity on f and suppose that it is continuous at every interior point of R except possibly at a finite number of points $P_1, P_2, \cdots, P_i, \cdots, P_n$. We shall refer to these exceptional points in the sequel as the points P_i. Our other condition on f now takes the form "about each interior point of R there exists a region M which is such that in its interior $\int (\partial f/\partial n) ds$ evaluated over every sphere which lies in M and does not pass through one of the P_i is zero." It is sufficient that if M contains one of the exceptional points, it contains only one.

That f is harmonic at every interior point P of R other than the P_i follows readily. About each of the P_i as center draw a small sphere S_i which does not contain P. Then the region bounded by the S_i and the boundary of R is a region of the type considered in Theorem I from which it follows that f is harmonic at P. It thus remains only to consider the nature of f in the neighborhood of any one of the P_i.

In a paper presented to the Society, October 31, 1925, the writer has shown that if a function is harmonic at every point in the deleted neighborhood of a point P it may be expressed in the form

$$c\frac{1}{r} + \Phi(x,y,z) + V(x,y,z)$$

in this neighborhood. In this expression c is a constant, r the distance from P to (x, y, z), V a function harmonic everywhere in the neighborhood of P as well as at P itself and Φ a function harmonic in the deleted neighborhood and such that it is either identically zero or else there exist modes of approach to P for which Φ will tend toward plus infinity and also modes of approach for which it will tend toward minus infinity; Φ also possesses the property that its integral over the surface of any sphere with P as center vanishes.

Consider now two spheres S_1 and S_2 with center P and radii r_1 and $r_2 < r_1$. Apply Green's formula to the functions Φ and $1/r - 1/r_1$ for the region bounded by S_1 and S_2 and we have

$$\int_{S_1S_2}\left\{\left(\frac{1}{r} - \frac{1}{r_1}\right)\frac{\partial\Phi}{\partial n} - \frac{\partial}{\partial n}\left(\frac{1}{r} - \frac{1}{r_1}\right)\Phi\right\}ds = 0,$$

where the normal derivatives are taken toward the interior of the region $S_1\,S_2$. Remembering that the integral of Φ over any sphere with center P is zero and since $1/r - 1/r_1$ is zero on S_1 and constant on S_2 we have from the above equation

$$\int_{S_2}\frac{\partial\Phi}{\partial n}\,ds = 0. \tag{3}$$

Since S_2 is any sphere interior to S_1 it follows that the integral of the normal derivative of Φ over any sphere with center P and radius less than r_1 vanishes. The same result could of course be obtained from the property $\int_S \Phi ds = 0$ by considering the continuity of Φ and using the theorem concerning the differentiation of a definite integral. In (3) because of the continuity of the first partial derivatives of Φ in the deleted neighborhood of P we may take the normal derivative *either* toward the interior or toward the exterior normal of S_2 throughout the region of integration. Hence if the function

$$f = c\frac{1}{r} + \Phi + V$$

is to be such that the integral of its normal derivative vanishes when taken over spheres in the neighborhood of P, the constant c must be zero. Conversely we have shown by the above argument that if c is zero $\int (\partial f/\partial n)ds$ will vanish when taken over every sphere with center P and of radius $r<r_1$. We may now state the following theorem.

THEOREM II. *Every function which satisfies the conditions of* § 2 *in a region R is harmonic at every interior point of R except possibly at the points P_i. In the neighborhood of each P_i, f is of the form $\Phi + V$. If $\Phi \equiv 0$, in the neighborhood of any P_i, P_i is at most a removable discontinuity. If $\Phi \not\equiv 0$, f will be harmonic in the deleted neighborhood of P and will be such that for certain modes of approach to P it will tend toward plus infinity and for other modes to minus infinity.*

It may be remarked in closing that although we have supposed $\int (\partial f/\partial n)ds$ to vanish only when taken over sufficiently small spheres with P_i as center it is now easy to prove that it will vanish when taken over any regular surface S in R which does not pass through one of the P_i. We need merely to surround each P_i in S by a sphere S_i lying entirely in S and use the fact that

$$\int \frac{\partial f}{\partial n} ds = 0 \qquad (4)$$

where the integral is taken over S and the spheres S_i. Then $\int_S (\partial f/\partial n)ds$ will vanish since the portion of (4) due to the S_i vanishes.

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