ON THE INTEGRO-DIFFERENTIAL EQUATION OF THE BÖCHER TYPE IN THREE-SPACE

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1. Introduction. Bôcher has shown* that if a function \( f(x, y) \) is continuous and has continuous first partial derivatives in a region \( R \) and satisfies the condition
\[
\int_{C} \frac{\partial f}{\partial n} ds = 0
\]
for every circle \( C \) lying entirely in \( R \), then \( f(x, y) \) is harmonic at each interior point of \( R \). Bôcher treats only functions in two variables and by a method which cannot be directly extended to three-space.

It is the purpose of the present note to show, by a simple modification of the second part of Bôcher's argument, that this result may at once be extended to three-space, and also to investigate the nature of the function \( f \) if Bôcher's condition of continuity is somewhat weakened. We shall treat explicitly functions in three variables only, but it will easily be seen that with a slight modification the statements of Theorem II are applicable to two-space as well.

THEOREM I. If a function \( f(x, y, z) \) is continuous, and has continuous first partial derivatives in a connected finite region \( R \), and is such that the surface integral \( \int_{S} (\partial f/\partial n)ds \) vanishes when taken over every sphere \( S \) lying in \( R \), then at each interior point of \( R \), \( f \) is harmonic; that is, it satisfies Laplace's equation
\[
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0
\]
at each interior point of \( R \).

In the preceding integral, as well as in what follows, the derivative \( df/dn \) is to be taken either toward the interior of \( S \), or toward the exterior of \( S \), throughout the region of integration. Let \( P \) be any interior point of \( R \) and consider two spheres \( S_1 \) and \( S_2 \) of radii \( r_1 \) and \( r_2 < r_1 \) with centers at \( P \). By hypothesis, we have

\[
\int_S \frac{\partial f}{\partial n} \, ds = 0;
\]

or, setting \( ds = r^2 d\omega \), where \( d\omega \) is the element of area on the unit sphere with center \( P \),

\[
\int_S \frac{\partial f}{\partial n} \, d\omega = 0.
\]

It follows that

\[
(1) \quad \int_{r_2}^{r_1} dr \int_S \frac{\partial f}{\partial n} \, d\omega = 0.
\]

Because of the continuity of \( f \) and its derivatives the order of integrations in the above integral may be inverted and we have

\[
(2) \quad \int_{S_1} f d\omega - \int_{S_2} f d\omega = 0.
\]

Let \( f(P) \) be the value of \( f \) at the point \( P \). Then since \( f \) is continuous at \( P \) we obtain from (2) by letting \( r_2 \) approach zero,

\[
f(P) = \frac{1}{4\pi} \int_{S_1} f d\omega = \frac{1}{4\pi r_1^2} \int_{S_1} f ds.
\]

We thus see that our function \( f \) possesses the so-called mean-value property, that is, its value at the center of any sphere is the mean of its values on the surface of the sphere.

Consider now the function \( F \) which takes the same values as \( f \) on \( S_1 \) and which is harmonic interior to \( S_1 \). This function exists and can be expressed as a Poisson integral. It is well known that \( F \) also possesses the mean-value prop-
erty and hence so also does the difference \( f - F \). But a continuous function having the mean value property in a closed region \( R \) must take its greatest and least values on the boundary of \( R \). Since the difference \( f - F \) is identically zero on \( S_1 \) it follows that it is zero everywhere within \( S_1 \) and hence \( f \) must be harmonic at \( P \) as was to be proved.

It is evident that the original hypothesis that

\[
\int \frac{\partial f}{\partial n} ds = 0
\]

about every sphere in \( R \) is unnecessarily broad. All that is needed in the above proof is that each point \( P \) may be surrounded by a region, no matter how small, which is such that the above integral vanishes when taken over every sphere lying entirely within it.

2. A More General Theorem. We shall now weaken the original condition of continuity on \( f \) and suppose that it is continuous at every interior point of \( R \) except possibly at a finite number of points \( P_1, P_2, \ldots, P_n, \ldots, P_w \). We shall refer to these exceptional points in the sequel as the points \( P_i \).

Our other condition on \( f \) now takes the form "about each interior point of \( R \) there exists a region \( M \) which is such that in its interior \( f (\partial f/\partial n) ds \) evaluated over every sphere which lies in \( M \) and does not pass through one of the \( P_i \) is zero." It is sufficient that if \( M \) contains one of the exceptional points, it contains only one.

That \( f \) is harmonic at every interior point \( P \) of \( R \) other than the \( P_i \) follows readily. About each of the \( P_i \) as center draw a small sphere \( S_i \) which does not contain \( P \). Then the region bounded by the \( S_i \) and the boundary of \( R \) is a region of the type considered in Theorem I from which it follows that \( f \) is harmonic at \( P \). It thus remains only to consider the nature of \( f \) in the neighborhood of any one of the \( P_i \).

In a paper presented to the Society, October 31, 1925, the writer has shown that if a function is harmonic at every point
in the deleted neighborhood of a point $P$ it may be expressed in the form
\[
\frac{1}{r} + \Phi(x, y, z) + V(x, y, z)
\]
in this neighborhood. In this expression $c$ is a constant, $r$ the distance from $P$ to $(x, y, z)$, $V$ a function harmonic everywhere in the neighborhood of $P$ as well as at $P$ itself and $\Phi$ a function harmonic in the deleted neighborhood and such that it is either identically zero or else there exist modes of approach to $P$ for which $\Phi$ will tend toward plus infinity and also modes of approach for which it will tend toward minus infinity; $\Phi$ also possesses the property that its integral over the surface of any sphere with $P$ as center vanishes.

Consider now two spheres $S_1$ and $S_2$ with center $P$ and radii $r_1$ and $r_2 < r_1$. Apply Green's formula to the functions $\Phi$ and $1/r - 1/r_1$ for the region bounded by $S_1$ and $S_2$ and we have
\[
\int_{S_1} \left\{ \left( \frac{1}{r} - \frac{1}{r_1} \right) \frac{\partial \Phi}{\partial n} - \frac{\partial}{\partial n} \left( \frac{1}{r} - \frac{1}{r_1} \right) \Phi \right\} ds = 0,
\]
where the normal derivatives are taken toward the interior of the region $S_1 S_2$. Remembering that the integral of $\Phi$ over any sphere with center $P$ is zero and since $1/r - 1/r_1$ is zero on $S_1$ and constant on $S_2$ we have from the above equation
\[
(3) \quad \int_{S_2} \frac{\partial \Phi}{\partial n} ds = 0.
\]
Since $S_2$ is any sphere interior to $S_1$ it follows that the integral of the normal derivative of $\Phi$ over any sphere with center $P$ and radius less than $r_1$ vanishes. The same result could of course be obtained from the property $\int_{S} \Phi ds = 0$ by considering the continuity of $\Phi$ and using the theorem concerning the differentiation of a definite integral. In (3) because of the continuity of the first partial derivatives of $\Phi$ in the deleted neighborhood of $P$ we may take the normal derivative \textit{either} toward the interior or toward the
exterior normal of $S_2$ throughout the region of integration. Hence if the function

$$f = \frac{c}{r} + \Phi + V$$

is to be such that the integral of its normal derivative vanishes when taken over spheres in the neighborhood of $P$, the constant $c$ must be zero. Conversely we have shown by the above argument that if $c$ is zero $\int (\partial f / \partial n) ds$ will vanish when taken over every sphere with center $P$ and of radius $r < r_1$. We may now state the following theorem.

**Theorem II.** Every function which satisfies the conditions of § 2 in a region $R$ is harmonic at every interior point of $R$ except possibly at the points $P_i$. In the neighborhood of each $P_i$, $f$ is of the form $\Phi + V$. If $\Phi \equiv 0$, in the neighborhood of any $P_i$, $P_i$ is at most a removable discontinuity. If $\Phi \not\equiv 0$, $f$ will be harmonic in the deleted neighborhood of $P$ and will be such that for certain modes of approach to $P$ it will tend toward plus infinity and for other modes to minus infinity.

It may be remarked in closing that although we have supposed $\int (\partial f / \partial n) ds$ to vanish only when taken over sufficiently small spheres with $P_i$ as center it is now easy to prove that it will vanish when taken over any regular surface $S$ in $R$ which does not pass through one of the $P_i$. We need merely to surround each $P_i$ in $S$ by a sphere $S_i$ lying entirely in $S$ and use the fact that

$$\int \frac{\partial f}{\partial n} ds = 0$$

where the integral is taken over $S$ and the spheres $S_i$. Then $\int_S (\partial f / \partial n) ds$ will vanish since the portion of (4) due to the $S_i$ vanishes.

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