TWO-WAY CONTINUOUS CURVES*

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A continuous curve M will be said to be a two-way continuous curve, or to be "two-way continuous," provided it is true that between every two points of M there exist in M at least two arcs neither of which is a subset of the other. A point P of a continuum M is a cut point of M provided it is true that the point set $M-P$ is not connected. Every point of a continuum M which is not a cut point of M will be called a non-cut point of M.

In a paper *Concerning continua in the plane*,† among other results, I have established the following theorems which will be used in the proofs given in this paper.

I. *If K denotes the set of all the cut points of a continuum M, then every bounded, closed, and connected subset of K is a continuous curve which contains no simple closed curve.*

II. *Every cut point of the boundary of a complementary domain of a bounded continuum M is a cut point also of M.*

III. *If K, H, and N, respectively, denote the set of all the cut points, end points,‡ and simple closed curves of a continuous curve M, then $K+H+N=M$.*

IV. *If N denotes the point set consisting of all the simple closed curves contained in a continuous curve M, then every connected subset of $M-N$ is arcwise connected.*

These results will be referred to by number as here listed. We shall now prove the following additional theorems.

* Presented to the Society, May 1, 1926.

† Recently submitted for publication in the TRANSACTIONS OF THIS SOCIETY.

‡ For a definition of this term see R. L. Wilder, *Concerning continuous curves*, FUNDAMENTA MATHEMATICAE, vol. 7 (1925), p. 358.

THEOREM 1. *In order that a continuous curve M should be two-way continuous it is necessary and sufficient that every simple continuous arc of M should contain a subarc which belongs to some single simple closed curve of M.*

THEOREM 2. *In order that a continuous curve M should be two-way continuous it is necessary and sufficient that every arc of M should contain a non-cut point of M.*

Proof. The condition is sufficient. Let A and B denote any two points of a continuous curve M which satisfies the condition. The curve M contains one arc t from A to B. And from our hypothesis it follows that t contains an interior point O which is a non-cut point of M. It follows from a theorem of R. L. Moore's* that $M-O$ contains an arc s from A to B. Since s does not contain the point O of t, it follows that $t\neq s$, and therefore, that M is two-way continuous.

The condition is also necessary. Let t denote any definite arc of a two-way continuous curve M. By Theorem 1, t contains a subarc s which belongs to some simple closed curve J of M. It is a consequence of a theorem of R. L. Moore's† that J contains not more than a countable number of cut points of M. Since s belongs to J and contains uncountably many points altogether, it follows that s, and hence also t, must contain at least one non-cut point of M.

THEOREM 3. *In order that a continuous curve M should be two-way continuous it is necessary and sufficient that the set K of all the cut points of M should contain no continuum.*

Proof. That the condition is sufficient is almost a direct consequence of Theorem 2. For, since by hypothesis K can contain no continuum, therefore it can contain no arc.

* *Concerning continuous curves in the plane*, MATHEMATISCHE ZEITSCHRIFT, vol. 15 (1922), pp. 254–260, Theorem 1.

† *Concerning the cut points of continuous curves and of other closed and connected point sets*, PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 9 (1923), pp. 101–106, Theorem B*.

Hence, every arc of M must contain a non-cut point of M, and by Theorem 2, M is two-way continuous. The condition is also necessary. For suppose the set K of all the cut points of a two-way continuous curve M contains a continuum H. Then by (I), H is a continuous curve. Hence, H contains at least one arc t. But by Theorem 2, t must contain at least one non-cut point of M. Thus the supposition that K contains a continuum leads to a contradiction.

THEOREM 4. *The boundary of every complementary domain of a two-way continuous curve is itself two-way continuous.*

Proof. Let M denote the boundary of a complementary domain of a two-way continuous curve K. Then M is a continuous curve.* Suppose, contrary to this theorem, that M is not two-way continuous. Then from Theorem 3 it follows that M must contain a continuum H every point of which is a cut point of M. But by (II), every cut point of M is a cut point also of K. And since K is two-way continuous, by Theorem 3, not every point of H can be a cut point of K. Thus the supposition that M is not two-way continuous leads to a contradiction.

THEOREM 5. *If N denotes the point set consisting of all the simple closed curves contained in a two-way continuous curve M, then $M-N$ is totally disconnected.*

Proof. Suppose $M-N$ contains a connected set L consisting of more than one point. Then from (III) and (IV) it readily follows that L contains an arc t every point of which is a cut point of M. But this is contrary to Theorem 2. It follows that $M-N$ is totally disconnected.

THEOREM 6. *The boundary M of a complementary domain of a two-way continuous curve is the sum of two mutually exclusive point sets N and H, where N is the sum of a countable*

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* Cf. Miss Torhorst, *Über den Rand der einfach zusammenhängenden ebenen Gebiete*, MATHEMATISCHE ZEITSCHRIFT, vol. 9 (1921), p. 64 (73).

number of simple closed curves no two of which have more than one point in common, and H is a totally disconnected set of points every one of which is a limit point of N and is either a cut point or an end point of M.

Proof. By Theorem 4, M is a two-way continuous curve. Let G denote the collection of all the simple closed curves contained in M. R. L. Wilder* has shown that G is countable and that no two curves of G have more than one point in common. Let N denote the point set obtained by adding together all the curves of the collection G. Then let H denote the point set $M-N$. Since M is two-way continuous, it readily follows that every point of H is a limit point of N. By Theorem 5, H is totally disconnected, and by (III), every point of H is either a cut point or an end point of M. Hence, the sets N and H satisfy all the conditions of Theorem 6.

THEOREM 7. *In order that the boundary M of a complementary domain D of a continuous curve should be two-way continuous it is necessary and sufficient that M should contain a point set K such that* (1) *$D+K$ is uniformly connected im kleinen, and* (2) *every arc, if there be any, which K' (K plus all the limit points of K) contains, contains a non-cut point of M.*

Proof. The condition is necessary. For let $K=M$. Clearly $D+K$ is uniformly connected im kleinen. And since K' is two-way continuous, it follows by Theorem 2 that every arc of K' contains a non-cut point of M. The condition is also sufficient. Let M denote the boundary of a complementary domain D of a continuous curve, and suppose that M contains a point set K satisfying conditions (1) and (2) in the statement of Theorem 7. Let A and B denote any two points of M. Now M contains one arc t from A to B. Either t is a subset of K' or it is not. If t is a subset of K', then by hypothesis t contains an interior

* Loc. cit., Theorem 4.

point O which is not a cut point of M. Then by a theorem of R. L. Moore's,* $M-O$ contains an arc from A to B which does not contain O, and which, therefore, is not a subset of t. Now if t is not a subset of K', then since K' is closed, it readily follows that t contains an arc s which contains no point of K'. Let X and Y denote the end points and O an interior point of s. Let C be a circle having O as center and not enclosing any point of K. Within C and on s there exist points E, U, W, and G in the order X, E, U, O, W, G, Y. And within C there exist arcs EFG and UVW having only their end points in common with s and such that if D_1 and D_2 denote the interiors of the closed curves $UVWOU$ and $EUOWGFE$ respectively, then D_1 and D_2 are mutually exclusive domains each of which lies within C. Now since $D+K$ is uniformly connected im kleinen, and C encloses no point of K, it readily follows that not both D_1 and D_2 can contain a subset of D which has O for a limit point. Hence, either D_1 or D_2 must contain a segment QST of an arc QST which has its end points on s in the order X, Q, O, T, Y and such that if R denotes the interior of the closed curve $QOTSQ$, then R contains no point whatever of $D+M$. Hence, R lies wholly in some complementary domain G of $D+M$. It follows from a theorem of R. L. Moore's† that the boundary J of G is a simple closed curve. The curve J contains the arc QOT of t. It follows that M contains an arc from A to B which does not contain the point O of t, and which, therefore, is not a subset of t. Hence, in any case, M contains two arcs from A to B neither of which is a subset of the other, and therefore M is two-way continuous.

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* *Concerning continuous curves in the plane*, loc. cit.

† *Concerning continuous curves in the plane*, loc. cit., Theorem 4.