THE OCTOBER MEETING IN NEW YORK

The two hundred fifty-first regular meeting of the Society was held at Columbia University, on Saturday, October 30, 1926, extending through the usual morning and afternoon sessions. The attendance included the following fifty-five members of the Society:

Alexander, W. L. Ayres, A. A. Bennett, Birkhoff, Bower, B. H. Camp, Carrié, Abraham Cohen, Doermann, Eisenhart, Fiske, Fite, Fort, Gehman, Gronwall, Hedlund, Himwich, Huber, Dunham Jackson, Joffe, R. A. Johnson, O. D. Kellogg, A. L. Kimball, B. F. Kimball, Koopman, Kormes, Lamson, Langman, Lefschetz, Littauer, Michal, Frank Morley, Richard Morris, Mullins, Pepper, Pfeiffer, Pierpont, Raudenbush, Raynor, Reddick, R. G. D. Richardson, Ritt, Seely, Siceloff, Smail, Virgil Snyder, M. H. Stone, Teach, T. Y. Thomas, Veblen, Wedderburn, Weida, Anna Pell Wheeler, Whittemore, W. A. Wilson.

The Secretary announced the election of the following eighteen persons to membership in the Society:

Miss Julia Wells Bower, Vassar College;
Mr. Edward John Braun, Wayland Academy;
Professor Loren G. Butler, Albany College;
Dr. Ralph Douglas Doner, Purdue University;
Mr. Herbert Pulse Evans, University of Wisconsin;
Mr. Joel S. Georges, University of Chicago;
Professor Roy French Graesser, University of Arizona;
Mr. Joshua V. Longenecker, Farmers and Bankers Life Insurance Company, Wichita;
Mr. Marcus Evans Mullings, University of Cincinnati;
Mr. Louis John Paradiso, Lehigh University;
Mr. Herbert Armond Perkins, Hampton Institute;
Mr. James Ellis Powell, Michigan Agricultural College;
Mr. Paul Klein Rees, Texas Technological College;
Mr. Warren Alonzo Rees, Texas Agricultural and Mechanical College;
Mr. Davis Payne Richardson, University of Chicago;
Mr. James Holmes Sturdivant, University of Texas;
Mr. Cecil Francis Whitaker, Macon, Ga.;
Professor Herman Zanstra, University of Washington.

Eleven applications for membership in the Society were received.

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The Board of Trustees at their meeting revised the budget and transacted other financial business.

At the meeting of the Council it was announced that the committee on the Josiah Willard Gibbs Lecture had selected Dr. H. B. Williams, Professor of Physiology, College of Physicians and Surgeons, Columbia University to give the fourth Gibbs Lecture in Philadelphia at the time of the Annual Meeting and that his subject is *Mathematics and the Biological Sciences.* President Birkhoff appointed as committee on arrangements for this lecture, Professors F. H. Safford (chairman) and J. A. Miller and Miss Marion Reilly.

It was announced that President Birkhoff had appointed Vice-President Evans to represent the Society at the semi-centennial celebration of the Agricultural and Mechanical College of Texas on October 14–16, 1926.

Professor A. B. Coble was invited to give a Colloquium in connection with the Summer Meeting in Amherst in 1928.

A list of nominations for trustees, officers, and other members of the Council was adopted unanimously and ordered printed on the ballot.

President Birkhoff presided at the morning session, relieved in the afternoon by Professor O. D. Kellogg.

Titles and abstracts of the papers read at this meeting follow below. The papers of Douglas, Ettlinger, Franklin, Graves, Hollcroft, Moore, Rice, Shaub, Weisner, Whyburn, and Wilson were read by title. Professor Drach was introduced by Professor Fite, Professor LaMer by Dr. Gronwall, and Mr. Serghiesco by Professor Ritt.

1. Dr. B. O. Koopman: *On analytic solutions of differential equations in the neighborhood of non-analytic singular points.*

This paper will appear in an early issue of this Bulletin.

2. Professor J. F. Ritt: *A factorization theory for functions* $\sum_{i=1}^{n} a_i e^{\alpha_i x}$.

When, in the expression $a_0 e^{\alpha_0 x} + \cdots + a_n e^{\alpha_n x}$, we allow n to assume all positive integral values, and the a's and α's all constant values, we obtain a class of functions which is closed with respect to multiplication. The problem thus arises of determining all representations of a given function

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of this class as a product of functions of this class. Simplicity of results is obtained, without loss of generality, by the assumption that $\alpha_0=0$, $a_0=1$. A function in which the α's all have rational ratios to one another is called simple. A function which is not the product of two functions is called *irreducible*. It is proved that every function can be expressed, in one and only one way, as a product of simple functions and of irreducible functions, the α's of every simple function having irrational ratios to those of every other.

3. Professor J. F. Ritt: *Simplification of Liouville's method in the theory of elementary functions.*

Liouville, about a century ago, studied the possibility of solving differential equations in finite terms. He was followed in this work by certain Scandinavian and Russian mathematicians. The present paper gives a much more powerful method than that of Liouville for handling questions of expressibility in finite terms. With the new method, the proofs of results already in the literature can be shortened from fifty to ninety percent. An illustration of the application of the method is contained in the Comptes Rendus for August 2, 1926.

4. Dr. B. F. Kimball: *Geodesics on a toroid.*

The surface studied is called a toroid, as it resembles a torus; it is, however, more general. The geodesics on this surface are described and classified. The conjugate points on each class of geodesics are also discussed, and some space is given to the description of the envelopes of the geodesics. The envelopes are investigated by calculating the second variation and then applying the theory of Lindeberg (Mathematische Annalen, vol. 59 (1904), p. 321). The method would seem to be a productive one.

5. Mr. S. Serghiesco: *Interpretation of the characteristic (in the Kronecker-Picard sense) of a particular system of four functions.*

Let $F_1(x, y, z)$, $F_2(x, y, z)$, $F_3(x, y, z)$, $F_4(x, y, z)$ be a system of four given continuous functions, $F_4(x, y, z)=0$ representing a certain closed surface S. It is known that the surface integral $I=(1/4\pi)\iint_S A\,dydz+B\,dzdx+C\,dxdy$, where A, B, C are special functions of $(F_1\cdots, F_{1x}', \cdots, F_{1y}'\cdots, F_{1z}'\cdots)$ is defined as the characteristic of F_1, F_2, F_3, and F_4. It is shown in this paper that when the four given functions are $F(x, y, z)$, $F_x'(x, y, z)$, $F_y'(x, y, z)$, $F_z'(x, y, x)$, the characteristic of these functions represents the total curvature, divided by 4π, of the whole closed surface S: $F(x, y, z)=0$. A geometrical verification is given through theorems of Gauss and definitions in differential geometry.

6. Dr. H. M. Gehman: *Concerning homogeneous plane continua.*

Mazurkiewicz has proved (Fundamenta Mathematicae, vol. 5 (1924), p. 137) that the only homogeneous plane continuous curves that exist

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are the simple closed curve, the open curve, and the plane. It is not known whether there exist plane continua which are not continuous curves. The following theorems give some information concerning the nature of such continua if they do exist. (1) If a homogeneous plane continuum contains the interior of any circle, it contains all points of the plane. (2) If one point of a homogeneous plane continuum M is a cut-point, then M is an open curve. (3) A homogeneous plane continuum which is an irreducible continuum between two points is indecomposable.

7. Mr. W. L. Ayres: *A new characterization of plane continuous curves.*

This paper will appear in full in an early issue of this Bulletin.

8. Mr. V. B. Teach: *On the properties of fields for the Lagrange problem in parametric form.*

For the problem of Lagrange in the calculus of variations, when stated in the parametric form in the usual manner, a field analogous to that of Mayer for the problem in non-parametric form is defined. Theorems analogous to those on the structure of a field for the problem in non-parametric form are then derived, and the construction of fields and some properties of transversal surfaces discussed.

9. Dr. Jesse Douglas (National Research Fellow): *The transversality relative to a surface* $\int F(x, y, z, y', z')dx =$ *minimum.*

This paper has appeared in the November-December number of this Bulletin.

10. Dr. Jesse Douglas (National Research Fellow): *A characteristic property of minimal surfaces.*

This paper has appeared in the November-December number of this Bulletin.

11. Professor R. L. Moore: *Concerning paths that do not disconnect a given continuous curve.*

It is shown that if, in a plane S, M is a bounded continuous curve, A and B are points of $S-M$ which do not lie in the same complementary domain of M, and K is the point set obtained by adding together all simple closed curves which lie in M and separate A from B, then (1) K is closed, and (2) there exists a simple continuous arc from A to B which does not disconnect M, that is to say, there exists an arc AB such that $M-(AB)\cdot M$ is connected.

12. Professor W. A. Wilson: *On the separation of a plane by two irreducible continua.*

This paper is a generalization of A. Rosenthal's results. It is shown that the conclusion of his principal theorem is valid if the two continua C_1 and

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C_2 have in common two point sets α and β, where $\alpha \cdot \beta = 0$, if each of the continua C_1 and C_2 is irreducible between any point a of α and any point b of β, and both C_1 and C_2 are decomposable or one of them is indecomposable and the other is not the union of two indecomposable continua. It is also shown that the frontier of each secondary region (Nebengebiet) is a part of some oscillatory set of C_1 or C_2 or is a part of the union of the oscillatory sets of C_1 and C_2 about a or b; consequently, such a frontier is either a continuum of condensation of $C_1 + C_2$, an indecomposable continuum, or the union of two indecomposable continua.

13. Professor H. J. Ettlinger: *Note on continuity in several variables.*

This paper appears in full in the present issue of this Bulletin.

14. Professor L. M. Graves (National Research Fellow): *On the existence of the absolute minimum in space problems of the calculus of variations.*

Tonelli in his recent *Fondamenti di Calcolo delle Variazioni* has developed new and powerful methods for treating the problem of the absolute minimum. The present paper applies these methods, with some necessary alterations, to the ordinary or non-parametric integrals $I_C = \int f(x, y, y')dx$ in space of $k+1$ dimensions. The classes of curves in which the minimum is sought are those called "complete" by Tonelli, and are so general as to include cases of fixed or variable end points, and problems of Lagrange with finite side conditions. A case of the latter would arise from application of Hamilton's principle to problems in dynamics with holonomic constraints.

15. Mr. W. M. Whyburn: *The Green's function for systems of differential equations whose coefficients contain a parameter.*

This paper treats a system of n differential equations together with n linear boundary conditions connecting the values of the functions at two points of the interval of definition. The coefficients are continuous functions of a parameter and summable functions of the independent variable, and finally are uniformly bounded. A theorem due to Bôcher (this Bulletin vol. 20 (1914), p. 1) on the reduction of the order of compatibility of a differential system is carried over, and this extended theorem is used to extend the author's previous results (Annals of Mathematics, (2), vol. 26 (1924), pp. 125–130) to more general systems. A matrix of $(n)^2$ Green's functions (Bounitzky, Journal de Mathématiques, (6), vol. 5 (1909), pp. 65–125) is set up, and the properties of these functions are discussed.

16. Dr. Louis Weisner (National Research Fellow): *A theorem concerning direct products.*

This paper appears in full in the present issue of this Bulletin.

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17. Mr. L. H. Rice: *Compounds of Cayley products of determinants of higher class.*

Continuing the general subject treated in previous papers (Journal of Mathematics and Physics, Massachusetts Institute of Technology, vol. 5, Nos. 1 and 4) which concerned both ordinary determinants and those of higher class, the writer shows that the mth compound of a Cayley product of two determinants of any classes and any signancies is the like Cayley product of the mth compounds of the factor determinants, and, as a corollary, that the same is true of the adjoint of a Cayley product. The paper will appear in volume 6 of the same Journal.

18. Professor T. R. Hollcroft: *Singularities of the Hessian.*

This paper appears in full in the present issue of this Bulletin.

19. Mr. H. C. Shaub: *Rational involutorial transformations in S_4 which leave invariant ∞^4 quadric varieties.*

By considering the (2, 1) correspondence between two spaces (x) and (x') of four dimensions for which a hyperplane in (x') has for image a quadric variety in (x), the author reduces the involutorial transformations which leave invariant ∞^2 quadratic varieties to four distinct types.

20. Professor Philip Franklin: *The classification of quadrics in euclidean three- and n-space, by means of covariants.*

Professor MacDuffee has recently given (American Mathematical Monthly, vol. 33, p. 243) the first complete classification of conics in the euclidean plane in terms of invariants and covariants. He uses the methods of the Lie theory. In this paper we give a complete classification of quadrics in n-space in terms of covariants. The system of covariants here used forms a Lie complete system, though our method of obtaining them is purely algebraic. The geometric significance of our results is also indicated.

21. Professor Philip Franklin: *The Simson lines of a triangle and the three-cusped hypocycloid.*

Steiner, in 1856, gave a discussion of the curve enveloped by the Simson lines of a triangle, and deduced many of its properties. He states (Werke, vol. 2, p. 642) that the curve is a three-cusped hypocycloid. While numerous analytical proofs of this have been given, the writer knows of no simple, direct synthetic proof in the literature. In this note such a proof is given. Certain related theorems are also obtained.

22. Professor G. D. Birkhoff: *On one-to-one transformations of surfaces.*

This paper deals with the structure of one-to-one analytic transformations T that do not possess an invariant area integral. The basis of the

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treatment consists in the use of the author's extended form of the last geometric theorem of Poincaré, and of certain notions applicable to such transformations which are contained in a paper about to appear in the Göttinger Nachrichten.

23. Professor James Pierpont: *Classification of quadric surfaces in hyperbolic space.*

J. L. Coolidge (Transactions of this Society, vol. 4 (1903)) and T. J. Bromwich (the same Transactions, vol. 6 (1905)) have shown how to classify quadric surfaces in hyperbolic space. Coolidge's method rests on the discussion by Clebsch and Lindemann of the relative position of two quadric surfaces, while Bromwich's makes use of elementary divisors. Both methods are very neat, but require a considerable knowledge of geometry or algebra. A more elementary method has been employed by P. D. Schwartz to classify conics (see abstract in this Bulletin, vol.32 (1926), p. 315). An extension of these principles to space leads to a classification of quadric surfaces.

24. Dr. C. M. Huber: *On complete systems of irrational invariants of associated point sets.*

The principal object of this paper is the determination of complete systems of invariants for sets of n points P_n^1 on a line. For $n=2p+2$, it is shown that all invariant products of the differences of the roots of the binary $(2p+2)$-ic are expressible in terms of elementary products linear in each root. The invariants for a set Q_{2p+2}^p defined by P_{2p+2}^1 are expressed in terms of the linear invariants for P_{2p+2}^1. For n odd it is conjectured that a complete system is made up of cyclic invariants. This is proved for n equal to 5 and 7, and a method is given for testing the validity of the conjectured theorem for any particular value of n. A mapping problem in connection with the cyclic invariants for P_5^1 and P_7^1 is discussed.

25. Professor Solomon Lefschetz: *Intersections and complete transformations of complexes and manifolds.* Second paper.

In this paper the author extends the results of his paper in the January, 1926, number of the Transactions of this Society to manifolds with a boundary when the topological behavior at all points of the latter is the same. It is found necessary to restrict somewhat the transformations considered. The results are surprisingly parallel to those for the no-boundary case.

26. Dr. M. H. Stone: *Expansions associated with a regular differential system of the second order.*

Associated with a differential system $u''+(\rho^2+g)u=0$, $0\leqq x\leqq 1$, $\alpha_i u^{(k_i)}(0)+\beta_i u^{(k_i)}(1)+\cdots=0$, $i=1, 2$, where g is a Lebesgue integral,

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and the boundary conditions are in normal form and regular, is a formal series expansion for any function integrable Lebesgue or Denjoy. The series for the same function in connection with two such systems are said to be equivalent on an interval (a, b) when the difference of partial sums of the two series is $o(1)$ uniformly on (a, b). It is shown here that a necessary and sufficient condition for equivalence on $(0, 1)$ of the series for an arbitrary summable function is that the boundary conditions of the two systems satisfy one of the following three relations: $k_1=\bar{k}_1=k_2=\bar{k}_2=1$; $k_1=\bar{k}_1=k_2=\bar{k}_2=0$; $k_1=\bar{k}_1=1$, $k_2=\bar{k}_2=0$, $\alpha_1\bar{\beta}_1-\bar{\alpha}_1\beta_1=0$, $\alpha_2\bar{\beta}_2-\bar{\alpha}_2\beta_2=0$. This condition is also shown necessary and sufficient for the equivalence on an interval (a, b) completely interior to $(0, 1)$ of the two series formed for an arbitrary function integrable in the sense of Denjoy. Finally, the first order Cesàro mean of the series of any function integrable in the sense of Denjoy is examined.

27. Dr. T. H. Gronwall and Professor V. K. LaMer: *The variation of the dielectric constant in the Debye-Hückel theory.*

In this paper a method is developed for determining the ion sizes and the variation of the dielectric constant with the concentration, from measurements of vapor pressure and freezing-point lowering in a solution of a strong electrolyte.

28. Dr. T. H. Gronwall: *A diophantine equation connected with the hydrogen spectrum.*

It is shown that in the Rydberg-Ritz formula for the wave length of a line in the hydrogen spectrum, the same wave length may correspond to two different sets of quantum numbers.

29. Professor Jules Drach: *Determination of Liouville's elements of length with algebraic integrals for the equation of geodesics.*

If $ds^2=4\lambda(u, v)\,du\,dv$ is an element of length in symmetric coordinates, the determination of the geodesics requires integration of the partial differential equation $pq=\lambda$, where $p=\partial z/\partial u$, $q=\partial z/\partial v$. In Liouville's case, where $\lambda=F(\alpha)-G(\beta)$, with $\alpha=u-v$, $\beta=u+v$, we know the quadratic integral $\phi=p^2+q^2+2pq(F+G)/(F-G)$; another integral, giving the general geodesic line, is ψ defined by $d\psi=d\alpha/(\phi+4F)^{1/2}+d\beta/(\phi+4G)^{1/2}$ and expressed in ϕ, α, β. The author wishes to determine all forms of F and G for which another integral f, rational in p and q, exists. In these cases a certain function of ψ is an algebraic function of ψ and f. For simplicity it is supposed that f is a polynomial in p, q.

R. G. D. Richardson,
Secretary.

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