

## A THEOREM CONCERNING DIRECT PRODUCTS\*

BY LOUIS WEISNER†

The theorem in question may be stated as follows. *A group of order  $mn$ , where  $m$  and  $n$  are relatively prime, in which every element whose order divides  $m$  is commutative with every element whose order divides  $n$ , is the direct product of two groups of orders  $m$  and  $n$ .*

Burnside‡ has proved the theorem for the special case in which either  $m$  or  $n$  is a power of a prime. Hence, to prove the theorem, we need only show that it is true for groups of order  $mn$  if it is true for groups of order  $<mn$ . To avoid trivial cases we assume that  $m$  and  $n > 1$ . We denote by  $m_1, m_2$ , and  $n_1, n_2$ , divisors  $> 1$  of  $m$  and  $n$  respectively.

If the group  $G$  contains an element of order  $m$ , let  $p$  be a prime factor of  $m$  and  $p^\alpha$  the highest power of  $p$  that divides  $m$ . Every element of  $G$  whose order divides  $p^\alpha$  is commutative with every element whose order is prime to  $p$ . It follows from the special case referred to, that  $G$  contains an invariant subgroup of order  $mn/p^\alpha$ , which contains an invariant subgroup of order  $n$ , as every element whose order divides  $m/p^\alpha$  is commutative with every element whose order divides  $n$ .

We suppose now that  $G$  contains no element of order  $m$ . The normaliser  $H$  of an element  $s$  of order  $m_1$  includes all the Sylow subgroups of  $G$  whose orders divide  $n$ , and is therefore of order  $m_2n$ , where  $m_2 \geq m_1$ . If  $m_2 < m$ ,  $H$  contains an invariant subgroup of order  $n$ . If  $m_2 = m$ ,  $s$  is invariant under  $G$ . An element  $t$  of  $G$  corresponding to an element  $t'$  of  $G/(s)$  of order  $n_1$  is of order  $n_1\mu$ , where  $\mu$  divides  $m_1$ . Hence  $G$  contains§ two elements  $t_1$  and  $s_1$  of orders  $n_1$  and  $\mu$  respectively, such that

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‡ W. Burnside, *Theory of Groups*, 2d edition, p. 327.

§ Loc. cit., p. 16.

$t = t_1 s_1$ . Since  $(t_1 s_1)^{n_1} = s_1^{n_1}$  is in  $(s)$ , and  $n_1$  is prime to the order of  $s_1$ ,  $s_1$  is a power of  $s$ . Thus  $t_1$ , as well as  $t$ , corresponds to  $t'$  in the isomorphism of  $G$  with  $G/(s)$ ; but  $t_1$  and  $t'$  are of the same order  $n_1$ . Every element of  $G/(s)$  whose order is a divisor of  $m/m_1$  corresponds to an element of  $G$  whose order is a divisor of  $m$ . It follows that  $t'$ , and hence every element of  $G/(s)$  whose order divides  $n$ , is commutative with every element whose order divides  $m/m_1$ . Hence  $G/(s)$ , being of order  $< mn$ , contains an invariant subgroup of order  $n$ . The corresponding subgroup of  $G$ , being of order  $m_1 n < mn$ , also contains an invariant subgroup of order  $n$ .

Thus in all cases  $G$  contains a subgroup of order  $n$ . Similarly  $G$  contains a subgroup of order  $m$ .  $G$  is evidently the direct product of these two subgroups.

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## A CUBIC CURVE CONNECTED WITH TWO TRIANGLES

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1. *Introduction.* If  $ABC$ ,  $XYZ$  are two triangles, a cubic curve  $\Gamma_3$  may be associated with them as follows.\* Let  $(PQ, RS)$  denote the point of intersection of the lines  $PQ, RS$ ; then  $\Gamma_3$  is the locus of a point  $O$  such that  $(OA, YZ)$ ,  $(OB, ZX)$ ,  $(OC, XY)$  are collinear and also the locus of a point  $O$  for which  $(OX, BC)$ ,  $(OY, CA)$ ,  $(OZ, AB)$  are collinear. In fact when one set of three points is collinear the other set of three is also collinear. Take  $ABC$  as triangle of reference and let the points  $X, Y, Z$  have coordinates  $(x_1, x_2, x_3)$ ,  $(y_1, y_2, y_3)$ ,  $(z_1, z_2, z_3)$  respectively, then if  $(\alpha, \beta, \gamma)$  are current coordinates

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\* H. Grassmann, *Die lineale Ausdehnungslehre*, 1844, p. 226. The corresponding quartic surface connected with two tetrahedra is mentioned by H. Fritz, Pr. Ludw. Gymn. Darmstadt [reference taken from *Jahrbuch der Fortschritte der Mathematik*, vol. 21 (1889), p. 725] and by C. M. Jessop, *Quartic Surfaces*, Cambridge, 1916, p. 189.