

ON SMALL DEFORMATIONS OF CURVES

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1. *Introduction.* This paper is concerned with small deformations of a single tortuous curve, of a family of curves on a given surface, and of a congruence of curves in space. In all cases, the displacement \mathbf{s} is supposed to be a small quantity of the first order, quantities of higher order being negligible.*

2. *Single Twisted Curve.* Consider first a given curve in space. The position vector \mathbf{r} of a point on the curve may be regarded as a function of the arc-length s of the curve, measured from a fixed point on it. Let \mathbf{t} , \mathbf{n} , \mathbf{b} be the unit tangent, principal normal and binormal. These are connected with the curvature κ and the torsion τ as in the Serret-Frenet formulas. Imagine a small deformation of the curve, such that the point of the curve originally at \mathbf{r} suffers a small displacement \mathbf{s} , its new position vector \mathbf{r}_1 being then

$$(1) \quad \mathbf{r}_1 = \mathbf{r} + \mathbf{s}.$$

Let a suffix unity be used to distinguish quantities belonging to the deformed curve, and let primes denote differentiations with respect to the arc-length s . Then the element $d\mathbf{r}_1$ of the deformed curve, corresponding to the element $d\mathbf{r}$ of the original, is given by $d\mathbf{r}_1 = d\mathbf{r} + d\mathbf{s}$, and its length ds_1 by

$$(ds_1)^2 = (d\mathbf{r}_1)^2 = (d\mathbf{r})^2 + 2d\mathbf{r} \cdot d\mathbf{s} = ds^2(1 + 2\mathbf{t} \cdot \mathbf{s}').$$

Consequently $ds_1 = ds(1 + \mathbf{t} \cdot \mathbf{s}')$.

The quantity $\mathbf{t} \cdot \mathbf{s}'$ represents the increase of length per unit length of the curve, or the extension of the curve at

*See also a paper by Perna, *Giornale di Matematiche*, vol. 36 (1898), pp. 286-299; and another by Salkowski, *Mathematische Annalen*, vol. 66 (1908), pp. 517-557.

the point considered. Let it be denoted by ϵ . Then

$$(2) \quad ds_1 = ds(1 + \epsilon) ; \quad ds = ds_1(1 - \epsilon).$$

The unit tangent t_1 to the deformed curve is given by

$$(3) \quad t_1 = \frac{d\mathbf{r}_1}{ds_1} = (1 - \epsilon)(\mathbf{r}' + \mathbf{s}') = (1 - \epsilon)\mathbf{t} + \mathbf{s}'.$$

Consequently, if κ_1 is the curvature and n_1 the unit principal normal for the new curve,

$$\kappa_1 n_1 = \frac{dt_1}{ds_1} = (1 - \epsilon) \frac{dt_1}{ds} = (1 - 2\epsilon)\kappa n - \epsilon't + s''.$$

On "squaring" both members, and neglecting small quantities of the second order, we have

$$\kappa_1^2 = (1 - 4\epsilon)\kappa^2 + 2\kappa n \cdot s'',$$

and therefore

$$(4) \quad \kappa_1 = (1 - 2\epsilon)\kappa + n \cdot s''.$$

Inserting this value in the above product $\kappa_1 n_1$ we find

$$(5) \quad \left\{ \begin{aligned} n_1 &= n - \frac{1}{\kappa} ((n \cdot s'')n + \epsilon't - s'') \\ &= n - (n \cdot s')t + \frac{1}{\kappa} (b \cdot s'')b. \end{aligned} \right.$$

The unit binormal b to the deformed curve is then given by

$$(6) \quad \left\{ \begin{aligned} b_1 &= t_1 \times n_1 = (1 - \epsilon)b + s' \times n - \frac{1}{\kappa} (b \cdot s'')n \\ &= b - (b \cdot s')t - \frac{1}{\kappa} (b \cdot s'')n. \end{aligned} \right.$$

The torsion τ_1 may then be found by differentiating the unit binormal, using the Serret-Frenet formulas. Thus on differentiating (6), and using the preceding results, we find on reduction

$$(7) \quad \tau_1 = (1 - \epsilon)\tau + \kappa b \cdot s' + \frac{d}{ds} \left(\frac{1}{\kappa} b \cdot s'' \right).$$

We have thus determined the geometric characteristics of the deformed curve in terms of those of the original curve and the small displacement \mathbf{s} . The rectangular trihedron, consisting of the unit tangent, the principal normal, and the binormal, undergoes a small rotation which is represented by the vector

$$(8) \quad \mathbf{R} = \frac{1}{\kappa} (\mathbf{b} \cdot \mathbf{s}'') \mathbf{t} - (\mathbf{b} \cdot \mathbf{s}') \mathbf{n} + (\mathbf{n} \cdot \mathbf{s}') \mathbf{b}$$

or

$$(9) \quad \mathbf{R} = \frac{1}{\kappa} (\mathbf{b} \cdot \mathbf{s}'') \mathbf{t} + \mathbf{t} \times \mathbf{s}'.$$

The coefficients of \mathbf{t} , \mathbf{n} , \mathbf{b} in (8) represent the small rotations of the trihedron about the tangent, the principal normal, and the binormal, respectively.

An *inextensional deformation* of the curve is one for which ϵ vanishes identically.* If \mathbf{s} is expressed in the form

$$\mathbf{s} = P\mathbf{t} + Q\mathbf{n} + R\mathbf{b},$$

the vanishing of $\mathbf{t} \cdot \mathbf{s}'$ gives $P' = \kappa Q$ as the necessary and sufficient condition for inextensional deformation.

3. *Family of Curves on a Surface.* Consider next a family of curves on a given surface. Suppose that the curves of the family suffer a small deformation such that they remain on the same surface. Then the displacement \mathbf{s} at any point is tangential to the surface, and is a function of two parameters that specify the point considered. On any one curve \mathbf{s} is a function of the arc-length s , and the formulas found above hold for the deformed curve.

Other results may be very neatly expressed in terms of the two-parametric differential invariants for the surface, introduced and examined by the author in a recent paper *On differential invariants in geometry of surfaces*.† If ϕ is the value of any function associated with the deformed curve at the point $\mathbf{r} + \mathbf{s}$, the new value of this function at

*This case has been considered in some detail by Sannia, *Rendiconti di Palermo*, vol. 21 (1906), pp. 229–256.

† *Quarterly Journal*, vol. 50 (1925), pp. 230–269.

the point \mathbf{r} originally occupied by this point of the curve is $\phi - \mathbf{s} \cdot \nabla \phi$, where ∇ is the two-parametric differential operator for the surface. Thus after the deformation the unit tangent $\bar{\mathbf{t}}$ to the new curve through the point \mathbf{r} is, by (3),

$$(10) \quad \begin{cases} \bar{\mathbf{t}} = \mathbf{t}_1 - \mathbf{s} \cdot \nabla \mathbf{t}_1 = (1 - \epsilon)\mathbf{t} + \mathbf{t} \cdot \nabla \mathbf{s} - \mathbf{s} \cdot \nabla \mathbf{t} \\ = (1 - \epsilon)\mathbf{t} + \text{curl}(\mathbf{s} \times \mathbf{t}) - \mathbf{s} \text{ div } \mathbf{t} + \mathbf{t} \text{ div } \mathbf{s}. \end{cases}$$

We have shown elsewhere* that the line of striction of a family of curves with unit tangent \mathbf{t} is given by $\text{div } \mathbf{t} = 0$. Hence the line of striction of the deformed family has for its equation $\text{div } \bar{\mathbf{t}} = 0$, which may be expressed in the form

$$(11) \quad (1 - \epsilon) \text{div } \mathbf{t} - \mathbf{t} \cdot \nabla \epsilon - \mathbf{s} \cdot \nabla \text{div } \mathbf{t} + \mathbf{t} \cdot \nabla \text{div } \mathbf{s} = 0,$$

since the divergence of the curl of the normal vector $\mathbf{s} \times \mathbf{t}$ vanishes identically.†

If the original family of curves is one of parallels, $\text{div } \mathbf{t}$ vanishes identically.‡ Hence *a family of parallels will remain parallels after the deformation provided*

$$(12) \quad \mathbf{t} \cdot \nabla(\epsilon - \text{div } \mathbf{s}) = 0.$$

The geodesic curvature of a curve of the original family§ is $\mathbf{n} \cdot \text{curl } \mathbf{t}$, and that of one of the deformed curves is

$$\mathbf{n} \cdot \text{curl} [(1 - \epsilon)\mathbf{t} + \mathbf{t} \cdot \nabla \mathbf{s} - \mathbf{s} \cdot \nabla \mathbf{t}].$$

If then the family of curves is a family of geodesics, it will remain so after the deformation provided

$$\mathbf{n} \cdot [\mathbf{t} \times \nabla \epsilon + \text{curl}(\mathbf{t} \cdot \nabla \mathbf{s} - \mathbf{s} \cdot \nabla \mathbf{t})] = 0.$$

Similarly, the original family will be *lines of curvature*|| if $\mathbf{t} \cdot \text{curl } \mathbf{t} = 0$; and they will remain lines of curvature after the change provided $\bar{\mathbf{t}} \cdot \text{curl } \bar{\mathbf{t}} = 0$.

* *Some new theorems in geometry of a surface*, § 2, *Mathematical Gazette*, vol. 13 (1926), pp. 1-6.

† *Quarterly Journal*, loc. cit., § 7.

‡ See a paper by the author *On families of curves and surfaces*, § 7, recently communicated to the *Messenger of Mathematics*.

§ *Quarterly Journal*, loc. cit., § 8.

|| *Mathematical Gazette*, loc. cit., § 4.

4. *Congruence of Curves in Space.* Consider finally a small deformation of a congruence of curves in three dimensions. On any one curve the displacement \mathbf{s} is a function of a single parameter; but on the congruence it is a point-function in space. Let ∇ now represent the three-parametric differential operator for three dimensions. Then the extension ϵ is $\epsilon = \mathbf{t} \cdot \mathbf{s}' = \mathbf{t} \cdot (\nabla \mathbf{s}) \cdot \mathbf{t}$, and, by the same argument as above, the unit tangent $\bar{\mathbf{t}}$ to the curve of the congruence which passes through the point \mathbf{r} after the deformation is $\bar{\mathbf{t}} = (1 - \epsilon)\mathbf{t} + \mathbf{t} \cdot \nabla \mathbf{s} - \mathbf{s} \cdot \nabla \mathbf{t}$, which may also be expressed in the same form as (10).

The surface of striction (or orthocentric surface) of a congruence with unit tangent \mathbf{t} is given by* $\text{div } \mathbf{t} = 0$. Hence the surface of striction of the deformed congruence has for its equation $\text{div } \bar{\mathbf{t}} = 0$, which may be expanded in the form $(1 - \epsilon) \text{div } \mathbf{t} - \mathbf{t} \cdot \nabla \epsilon - \mathbf{s} \cdot \nabla \text{div } \mathbf{t} + \mathbf{t} \cdot \nabla \text{div } \mathbf{s} = 0$, since the divergence of the curl now vanishes identically. The limit surface† of the congruence before deformation is

$$(13) \quad \text{div } (\mathbf{t} \text{ div } \mathbf{t} - \mathbf{t} \cdot \nabla \mathbf{t}) = 0, \dagger$$

and that of the deformed congruence is given by a similar equation in which $\bar{\mathbf{t}}$ takes the place of \mathbf{t} . The deformed curves will constitute a normal congruence provided we have

$$\bar{\mathbf{t}} \cdot \text{curl } \bar{\mathbf{t}} = 0,$$

and they will constitute an isometric congruence if, in addition, $\bar{\mathbf{t}}$ satisfies the equation§

$$(14) \quad \text{curl } (\bar{\mathbf{t}} \text{ div } \bar{\mathbf{t}} - \bar{\mathbf{t}} \cdot \nabla \bar{\mathbf{t}}) = 0$$

which may be expanded in terms of \mathbf{t} and \mathbf{s} .

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* See a paper by the author *On congruences of curves*, § 6, recently communicated to the Tôhoku Mathematical Journal.

† Ibid., § 4.

‡ This equation for the limit surface of a curvilinear congruence was first given by the author in a paper *On isometric systems of curves and surfaces*, recently communicated to the American Journal.

§ Ibid., § 6.