INTEGERS AS TERNARY QUADRATIC FORMS

INTEGERS REPRESENTED BY POSITIVE TERNARY QUADRATIC FORMS*

BY L. E. DICKSON

1. Introduction. Dirichlet† proved by the method of §2 the following two theorems:

**Theorem I.** $A = x^2 + y^2 + z^2$ represents exclusively all positive integers not of the form $4^k(8n + 7)$.

**Theorem II.** $B = x^2 + y^2 + 3z^2$ represents every positive integer not divisible by 3.

Without giving any details, he stated that like considerations applied to the representation of multiples of 3 by $B$. But the latter problem is much more difficult and no treatment has since been published; it is solved below by two methods.

Ramanujan‡ readily found all sets of positive integers $a, b, c, d$ such that every positive integer can be expressed in the form $ax^2 + by^2 + cz^2 + du^2$. He made use of the forms of numbers representable by

- $A, B, C = x^2 + y^2 + 2z^2$,
- $D = x^2 + 2y^2 + 2z^2$,
- $E = x^2 + 2y^2 + 3z^2$,
- $F = x^2 + 2y^2 + 4z^2$,
- $G = x^2 + 2y^2 + 5z^2$.

He gave no proofs for these forms and doubtless obtained his results empirically. We shall give a complete theory for these forms. These cases indicate clearly methods of procedure for any similar form.

For a new theorem on forms in $n$ variables, see §9.

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* Presented to the Society, December 31, 1926.
‡ Proceedings of the Cambridge Philosophical Society, vol. 19 (1916–19), pp. 11–15. He overlooked the fact that $x^2 + 2y^2 + 5z^2 + 5u^2$ fails to represent 15.
2. **The Form B.** Let $B$ represent a multiple of 3. Since $-1$ is a quadratic non-residue of 3, $x$ and $y$ must be multiples of 3. Thus $B = 3\beta, \beta = 3X^2 + 3Y^2 + z^2$. Since $\beta \equiv 0$ or 1 (mod 3), $\beta$ represents no integer $3n+2$. If $\beta$ is divisible by 3, $z$ is divisible by 3 and $B$ is the product of a like form by 9. We shall prove that $\beta$ represents every positive integer $3n+1$. These results and Theorem II give

**Theorem III.** $x^2 + y^2 + 3z^2$ represents exclusively all positive integers not of the form $9k(9n+6)$.

We shall change the notation from $\beta$ to $f$ and employ the fact that the only reduced positive ternary forms of Hessian 9 are

- $f = x^2 + 3y^2 + 3z^2$,
- $g = x^2 + y^2 + 9z^2$,
- $h = x^2 + 2y^2 + 5z^2 - 2yz$,
- $l = 2x^2 + 2y^2 + 3z^2 - 2xy$.

No one of $g, h, l$ represents an integer $8m+7$. For $g$ this follows from Theorem 1, since $g \equiv A$ (mod 8). Suppose $l \equiv 7$ (mod 8). Then $z$ is odd, $2s \equiv 4$ (mod 8), where $s = x^2 + y^2 - xy$. Thus $s$ is even and $(1+x)(1+y) \equiv 1$ (mod 2), $x$ and $y$ are even, and $s \equiv 0$ (mod 4), a contradiction. Finally, let $h \equiv 7$ (mod 8). If $y$ is even, $h \equiv x^2 + z^2 \equiv 3$ (mod 4). Hence $y$ is odd and

\[ 3 \equiv h = x^2 + (z-1)^2 + 1 \pmod{4}, \]

so that $x$ and $z-1$ are odd. Write $z = 2Z$. Then $h \equiv 3 + 4Z (Z-1) \equiv 3$ (mod 8).

Consider the ternary form lacking the term $xy$:

\[ \phi = ax^2 + by^2 + cz^2 + 2ryz + 2szx. \]

Its Hessian $H$ is $a\Delta - bs^2$, where $\Delta = bc - r^2$. Take $H = 9, s = 1, \Delta = 24t, t = 6k+1$. Then $b = 3\beta, \beta = 8at - 3$. If $a$ is not divisible by 3, $\beta = 48ak + 8a - 3$ is a linear function of $k$ with relatively prime coefficients and hence represents an infinitude of primes.

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* Eisenstein, Journal für Mathematik, vol. 41 (1851), p. 169. By the Hessian $H$ of $\phi$ we mean the determinant whose elements are the halves of the second partial derivatives of $\phi$ with respect to $x, y, z$. Eisenstein called $-H$ the determinant of $\phi$. The facts borrowed in this paper from Eisenstein's table have been verified independently by the writer.
Take \( a = 3n + 1 \). Then \( \beta \equiv -1(\text{mod} 6) \),

\[
\begin{align*}
\left( \frac{-3}{\beta} \right) &= -1, \\
\left( \frac{2}{\beta} \right) &= -1,
\end{align*}
\]

(2) \[
\left( \frac{t}{\beta} \right) = \left( \frac{\beta}{t} \right) = \left( \frac{-3}{t} \right) = 1, \quad \left( \frac{-\Delta}{\beta} \right) = 1.
\]

Hence \( \omega^2 = -\Delta(\text{mod} \beta) \) is solvable. We can choose a multiple \( r \) of 3 such that \( r \equiv \omega(\text{mod} \beta) \). Then \((\Delta + r^2)/b\) is an integer \( c \). Since \( \phi \) represents \( b \equiv 7(\text{mod} 8) \), it is equivalent to no one of \( g, h, l \) and hence is equivalent to \( f \). Thus \( f \) represents every \( a = 3n + 1 \).

**Theorem IV.** \( x^3 + 3y^2 + 3z^2 \) represents exclusively all positive integers not of the form \( 9^k(3n+2) \).

This theory for \( B \) made use of forms of the larger Hessian \( 9 \). We shall next show how to deduce a theory making use only of forms having the same Hessian 3 as \( B \).

3. **A New Theory for \( B \).** We proved that \( f \) is equivalent to a form (1) having \( a = 3n + 1, \ b = 3\beta, \ r = 3\rho, \ s = 1 \), where \( \beta = 8\alpha t - 3 \) is a prime. In \( 9 = H = a(bc - r^2) - 3\beta \), replace \( \beta \) by its value. Thence \( c = (8t + 3\rho^2)/\beta \equiv 1(\text{mod} 3), \ c = 1 + 3\gamma \). In (1) replace \( x \) by \( X - z \). We get

\[
\psi = aX^2 - 6nXz + 3(n + \gamma)z^2 + 3\beta y^2 + 6\rho yz.
\]

Write \( 3z = X, \ 3y = Y \). Then

\[
3\psi = 3aX^2 - 6nXZ + (n + \gamma)Z^2 + \beta Y^2 + 2\rho YZ
\]

is equivalent to \( 3f = 3x^2 + Y^2 + Z^2 \) and hence to \( B \). In \( 3\psi \), replace \( a \) by \( \alpha \), \( \beta \) by \( b \), \( \rho \) by \( r \). We conclude that (1) is equivalent to \( B \) if

\[
(3) \quad a = 3\alpha, \quad \alpha = 3n + 1, \quad b = 8\alpha t - 3, \quad t = 6k + 1, \quad s = -3n = 1 - \alpha.
\]

We shall now give a direct proof that there exists a form (1) of Hessian 3 which satisfies conditions (3) and is equivalent to \( B \). In \( H = 3 \), replace \( a \) and \( s \) by their values in (3). We get

\[
b + 3 + 3\alpha r^2 - abP = 0, \quad P = 3c + 2 - \alpha.
\]
Replace the first term $b$ by its value in (3), and cancel $\alpha$. We get

$$8t + 3r^2 - bP = 0, \quad -24t \equiv (3r)^2 \pmod{b}. \tag{4}$$

This congruence is solvable by (2) with $\beta$ replaced by $b$. By (4), $8t(1-P)\equiv 0, P \equiv 1 \pmod{3}$. Hence the value of $c$ determined by $P$ is an integer. The only two reduced forms of Hessian 3 are $B$ and $\chi = x^2 + 2\sigma$, where $\sigma = y^2 + yz + z^2$. Suppose $\chi \equiv 5 \pmod{8}$. Then $x$ is odd and $\sigma \equiv 2 \pmod{4}$. Thus

$$(1 + y)(1 + z) \equiv 1, \quad y \equiv z \equiv 0 \pmod{2}, \quad \sigma \equiv 0 \pmod{4}.$$

This contradiction shows that $b$ is not represented by $\chi$. Since (1) represents $b$, it is not equivalent to $\chi$ and hence is equivalent to $B$. Thus $B$ as well as $\phi$ represents $a = 3\alpha$. This completes the new proof of Theorem III by using only forms of Hessian 3. The numbers represented by $\chi$ are given by Theorem XI.

4. The Form $C = x^2 + y^2 + 2z^2$. By Theorem I, $A$ represents every positive $4k + 2$. Then just two of $x$, $y$, $z$ are odd, say $x$ and $y$, while $z = 2Z$. Then $x = X + Y, y = X - Y$ determine integers $X$ and $Y$. Hence

$$X^2 + Y^2 + 2Z^2 = 2k + 1,$$

so that $C$ represents all positive odd integers.† If

$$m \neq 4^k(8n + 7), m = x^2 + y^2 + z^2,$$

by Theorem I. Hence

$$2m = (X + Y)^2 + (X - Y)^2 + 2z^2.$$

Conversely, if $C$ is even, it is of the latter form.

**Theorem V.** $C$ represents exclusively all positive integers not of the form $4^k(16n + 14)$.

5. The Form $D = x^2 + 2y^2 + 2z^2$. If $m$ is odd and $\neq 8n + 7, m = x^2 + Y^2 + Z^2$ by Theorem I. Then $x + Y + Z$ is odd. Permuting, we may take $x$ odd, and write $Y + Z = 2y, Y - Z = 2z$.

* If we apply the method of §2 when $H = 3$ and hence take $s = 1, b = 3\beta$, where $\beta$ is a prime $\equiv -1 \pmod{8}$, we find that it fails for all choices of $\Delta$.

†Lebesque, Journal de Mathématiques, (2), vol. 2 (1857), p. 149, gave a long proof by the method of §2.
Then \( m = D \). Next, let \( m = 2r \) be any even integer not of the form \( 4^k(8n+7) \). Then \( r \neq 4^k(16n+14) \). By Theorem V, \( r \) is represented by \( C \). Then \( m = 2r \) is represented by \( D \) with \( x \) even.

**Theorem VI.** \( D \) represents exclusively* all positive integers not of the form \( 4^k(8n+7) \).

6. *The Form \( F = x^2 + 2y^2 + 4z^2 \).* Every odd integer is represented by \( C \) with \( x+y \) odd, whence one of \( x \) and \( y \) is even. Any integer \( \neq 4^k(8n+7) \) is represented by \( D \), and \( 2D = (2y)^2 + 2x^2 + 4z^2 \).

**Theorem VII.** \( F \) represents exclusively† all positive integers not of the form \( 4^k(16n+14) \).

The simple methods used in proving Theorems V–VII apply also to \( x^2 + 2ry + 2sz \) when \( r \) and \( s \) are both \( \leq 3 \), and when \( r = 1 \) or \( 3, s = 4 \).

7. *The Form \( G = x^2 + 2y^2 + 5z^2 \).* The only reduced forms of Hessian 10 are \( G, J = x^2 + y^2 + 10z^2, K = 2x^2 + 2y^2 + 3z^2 + 2xz, \) and \( L = 2x^2 + 2y^2 + 4z^2 + 2yz + 2xz + 2xy \). Neither \( J \) nor \( K \) represents a number of the form \( 2(8n+3) \). For, if \( K \) is even, \( z = 2Z, K = 2M, M = (x^2 + y^2 + 5Z^2) \), where \( X = x + Z \). Since \( M \) is congruent to a sum of three squares modulo 4, it is congruent to 3 if and only if each square is odd, and then \( M \equiv 7 \) (mod 8). If \( J \) is even, \( x = y + 2t, J = 2N, N = (y + t)^2 + t^2 + 5z^2 \not\equiv 3 \) (mod 8).

We now apply the method of §2 to prove that \( G \) represents every positive integer prime to 10. Take \( \Delta = 16k, k = 10l \pm 3 \). Then \( b = 2\beta \), where \( \beta = 8ka - 5 \) represents infinitely many primes. Now

\[
\left( \frac{-\beta}{k} \right) = \left( \frac{5}{k} \right) = \left( \frac{k}{5} \right) = -1,
\]

\[
\left( \frac{-\Delta}{\beta} \right) = -\left( \frac{k}{\beta} \right) = -\left( \frac{-\beta}{k} \right) = 1.
\]

Since (1) represents \( b \), which is of the form \( 2(8n+3) \), it is not equivalent to \( J \) or \( K \). Since it represents the odd \( a \), it is not

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* \( D = x^2 + (y+z)^2 + (y-x)^2 \not\equiv 4^k(8n+7) \) by Theorem I.
† If \( F \) is even, \( x \) is even and \( F \) is the double of a form \( D \).
equivalent to \( L \). Hence (1) is equivalent to \( G \), which therefore represents \( a \).

If \( G \) represents a multiple of 5, it is the product of 5 by 
\[
g = 5X^2 + 10Y^2 + z^2,
\]
whence \( G \) represents no \( 5(5n \pm 2) \). Also, 
\( g \) is divisible by 5 only when \( z \) is. Thus \( G \) is divisible by 25 only when it is a product of 25 by a form like \( G \).

To prove* that \( G \) represents every \( 5a \) if \( a = 5n \pm 1 \) is odd, employ (1) with \( a = 5\alpha \), \( b = 2\beta \), \( \beta = 8\alpha t - 5 \), \( r = 2p \), \( s = 1 \mp \alpha \). The Hessian of (1) is 10 if 
\[
\beta + 5 + 10\alpha \rho^2 - \alpha \beta P = 0, \quad P = 5c \pm 2 - \alpha.
\]
Take \( t \) prime to 10 and replace the first \( \beta \) by its value. Thus 
\[
8t + 10\rho^2 - \beta P = 0, \quad P \equiv \pm 1 \pmod{5}.
\]
Hence \( P \) yields an integral value for \( c \). Also,
\[
\left( \frac{t}{\beta} \right) = \left( \frac{-\beta}{t} \right) = \left( \frac{5}{t} \right),
\]
\[
\left( \frac{5}{\beta} \right) = \left( \frac{\beta}{5} \right) = \left( \frac{\pm 8t}{5} \right) = - \left( \frac{t}{5} \right),
\]
\[
\left( \frac{-80t}{\beta} \right) = - \left( \frac{5t}{\beta} \right) = 1.
\]

Next, if \( G \) is even, \( x = z + 2w \) and \( G = 2T \), where 
\[
T = y^2 + 2w^2 + 2wz + 3s^2, \quad S = x^2 + y^2 + 5z^2
\]
are the only reduced forms of Hessian 5. Every positive integer \( a \) prime to 5 is represented by \( T \). Take \( \Delta = 8k, k = 10m \pm 1 \). Then \( b = a\Delta - 5 \) represents an infinitude of primes, and
\[
\left( \frac{-2}{b} \right) = 1, \quad \left( \frac{-\Delta}{b} \right) = \left( \frac{k}{b} \right) = \left( \frac{-b}{k} \right) = \left( \frac{5}{k} \right) = \left( \frac{k}{5} \right) = 1.
\]
Now \( b \equiv 3 \pmod{8} \) is not represented by \( S \), as proved for \( M \). Hence (1) is equivalent to \( T \) and not \( S \). Thus \( T \) represents \( a \).

* Or we may use the method of §2. Of the ten properly primitive reduced forms of Hessian 50, all except \( g \) fail to represent numbers \( \equiv 14 \pmod{16} \). To prove that \( g \) represents \( \alpha = 5n \pm 1 \) when odd, take \( \Delta = 80t \), whence \( b = 10\beta, \beta = 8at - 5 \); apply (5). From this proof was reconstructed the shorter one in the text.
We saw that $2T = G$ represents no $5(5m \pm 2)$. Thus $T$ represents no $5(5n \pm 1)$. To prove that $T$ represents every $5\alpha$, where $\alpha = 5n \pm 2$, employ (1) with $a = 5\alpha$, $s = 1 \pm 2\alpha$. Its Hessian is 5 if

$$b + 5 + 5\alpha^2 - abP = 0, \quad P = 5c - 4\alpha \pm 4.$$  

Replace the first term $b$ by $8t\alpha - 5$, where $t$ is prime to 10. Thus $8t + 5r^2 - bP = 0$. Hence $8t(1 \mp 2P) = 0$, $P \equiv \pm 3(\text{mod } 5)$, and the resulting value of $c$ is an integer. Also

$$\left(\frac{5}{b}\right) = \left(\frac{b}{5}\right) = \left(\frac{\pm 16t}{5}\right) = \left(\frac{t}{5}\right),$$

$$\left(\frac{t}{b}\right) = \left(\frac{-b}{t}\right) = \left(\frac{5}{t}\right), \quad \left(\frac{-40t}{b}\right) = 1.$$  

We have now proved the two theorems:

**Theorem VIII.** $T$ represents exclusively all $\not\equiv 25k(25m \pm 5)$.

**Theorem IX.** $G$ represents exclusively all $\not\equiv 25k(25n \pm 10)$.

8. *The Form $E = x^2 + 2y^2 + 3z^2$. We shall outline a proof of the following theorem.*

**Theorem X.** $E$ represents exclusively all $\not\equiv 4^k(16n + 10)$.

The only reduced forms of Hessian 6 are $E$ and

$$Q = x^2 + y^2 + 6z^2, \quad R = 2x^2 + 2y^2 + 2z^2 + 2xy.$$  

To prove that $E$ represents every positive odd integer $a$, take $\Delta = 9k$, $k = 8t + 3$. Then $b = 3\beta$, where $\beta$ represents primes. Also $(-\Delta/\beta) = 1$. The resulting form (1) represents the odd $a$ and hence is not equivalent to $R$. Since it represents $b = 3(3n + 1)$, it is not equivalent to $Q$. For, if $Q$ is divisible by 3, both $x$ and $y$ are.

If $E$ is even, then $x = x + 2t$ and $E = 2U$,

$$U = y^2 + 2z^2 + 2st + t^2$$

and conversely. In place of $U$ we employ the like form $\chi$ of §3. To show that $\chi$ represents $a = 2\alpha$ when $\alpha$ is odd, take $\Delta = 9k$, $k = 4t + 1$. Then $b = a\Delta - 3 \equiv 6(\text{mod } 9)$ is not represented by the remaining reduced form $B$ of Hessian 3 (Theorem III). Also $b = 3\beta$, $(-\Delta/\beta) = +1$.  

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Finally, $\chi$ represents every positive odd integer $a \not\equiv 5(\text{mod } 8)$. Write $\alpha = \frac{1}{2}(3a - 1)$ and take $\Delta = 9k$, $k = 2h + 1$. Then $b = 6q$, $q = 3ah + \alpha$. If $a = 8A + 1$, take $h = 4t$, $t$ odd. Then $q = 12Ak + 12t + 1$,

$\left(\frac{-\Delta}{q}\right) = \left(\frac{k}{q}\right) = \left(\frac{q}{k}\right) = \left(\frac{12t + 1}{k}\right) = \left(\frac{4t}{k}\right) = \left(\frac{k}{t}\right) = 1$.

If $a = 8A + 3$, take $h = 4t + 1$. If $a = 8A + 7$, take $h = 4t + 1$. In each case $(-\Delta/q) = 1$. In all three cases, $q$ represents an infinitude of primes.

**Theorem XI.** $x^2 + 2y^2 + 2yz + 2z^2$ represents exclusively all positive integers not of the form $4^k(8n + 5)$.

9. **Forms in n Variables.** By a simple modification of Ramanujan's determination of quaternary forms which represent all positive integers, we readily prove*

**Theorem XII.** If, for $n \geq 5$, $f = a_1x_1^2 + \cdots + a_nx_n^2$ represents all positive integers, while no sum of fewer than $n$ terms of $f$ represents all positive integers, then $n = 5$ and

$f = x^2 + 2y^2 + 5z^2 + 5u^2 + ev^2$, \hspace{0.5cm} (e = 5, 11, 12, 13, 14, 15),

and these six forms $f$ actually have the property stated.

After this paper was in type, I saw that J. G. A. Arndt gave† the Dirichlet type of proof which appears in §2 above, but not the improved new proof of §3. For the form $G$ of §7, he treats only numbers not divisible by 5.

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*Note to appear in Proceedings of the National Academy.

†Ueber die Darstellung ganzer Zahlen als Summen von sieben Kuben, Dissertation, Göttingen, 1925.