A CLASS OF TRANSCENDENTAL NUMBERS

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The purpose of this short note is to determine a class of transcendental numbers by considerations relating to series of powers. I shall give here a theorem quite different from those that I have given previously.* I shall prove the following theorem.

THEOREM I. *Let* $\sum a_n x^n = F(x)$ *be a power series having singularities only within the circle with center at the point* 1 *and radius* $1/(K-1)$, K *an integer, there being at least one essential singularity within the circle, not at the center.*

If there exists a number ζ *and an integer* $K' < \zeta$ *such that the quantities*

$$K'^n\left(a_n + \frac{1}{\zeta^{n+1}}\right) = N_n \qquad (n = 1, 2, \cdots)$$

are integers, then ζ *is a transcendental number.*

I have given elsewhere the following theorem.†

THEOREM II. *If the series* $\sum b_n x^n = f(x)$ *represents a function having singularities only within the circle with center at* 1 *and radius* $1/(K_1 - 1)$, *where* K_1 *is an integer, and if the quantities*

$$b_n K_1^n = N_n' \tag{1}$$

are integers, then we have

$$f(x) = \frac{P(x)}{(1 - x)^h},$$

in which h *is an integer and* $P(x)$ *a polynomial.*

From this theorem, I then proved the following theorem.

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* Comptes Rendus, Feb. 1, 1926; Journal de Mathématiques, 1926.
† Journal de Mathématiques, 1926.

THEOREM III. *If a series* $\sum c_n x^n = \phi(x)$ *has at least one singular point distinct from the point* 1, *and if an integer* K_2 *exists so that the quantities*

$$K_2^n c_n = N_n^{(2)}, \qquad (n = 1, 2, \cdots)$$

are also integers, then, if we denote by x_0 *the argument of the singularity of* $\phi(x)$ *which is farthest from the point* 1, *we shall have*

$$|x_0 - 1| > \frac{1}{K - 1}. \tag{2}$$

In these theorems, the point at infinity was counted with the other points; that is, in Theorem II, we assumed the point at infinity to be a regular point for $\sum b_n x^n$. Likewise in Theorem III, if the point at infinity happens to be a singular point, the inequality (2) is devoid of meaning. But we can now generalize Theorem II by adding that if the point at infinity is a pole for $f(x)$, and if the other singularities be within the circle above mentioned, we shall still have

$$f(x) = \frac{P(x)}{(1 - x)^h},$$

if the property (1) is satisfied.

The proof in this case will be almost the same as for II*. We will only remark that Hurwitz's theorem can be generalized to series of the form

$$f(x) = \sum_0^\infty \frac{\alpha_n}{x^{n-m}}, \quad \phi(x) = \sum_0^\infty \frac{\beta_n}{x^{n-m}},$$

viz., that the series

$$\frac{A_m}{x^m} + \cdots + \frac{A}{x} + A_0$$
$$+ \sum \frac{\alpha_{n+m}\beta_{m+1} - C_n' \alpha_{n+m-1}\beta_{m+1} + \cdots + (-1)^n \beta_{m+n}\alpha_{n+1}}{x^n}$$

* See Journal de Mathématiques, 1926.

has as singularities, besides the point at infinity, only points of the form $\alpha+\beta$, where α is a singular point of $f(x)$ and β a singular point of $\phi(x)$.

A. We can therefore see that if the series $\sum a_n x^n$ is such that there exists an integer K so that the quantities $a_n K^n$ are integers, then if $P(x)$ is a polynomial with integral coefficients, the function

$$P(x) \sum a_n x^n = \sum b_n x^n$$

is such that there exists an integer $K_1 \leqq K$, so that the quantities $K_1^n b_n$ are integers. This results immediately from the form of the coefficients b_n:

$$b_n = a_n A_0 + a_{n-1} A_1 + \cdots + a_{n-p} A_p,$$

where

$$P(x) = A_0 + A_1 x + \cdots + A_p x^p.$$

We write then, returning to the theorem to be proved,

$$\sum c_n x^n = \sum \left(a_n + \frac{1}{\zeta^n} \right) x^n = F(x) + \frac{1}{\zeta - x} \cdot$$

If we suppose that ζ is an algebraic number, there will be a polynomial with integral coefficients such that $P(x)/(\zeta - x)$ will be regular in the entire plane except for the point at infinity. Hence the function $\theta(x) = P(x) \cdot F(x)$ has singularities only within the circle with center at the point 1 and radius $1/(K'-1)$, except for a pole at infinity.

This results from the remark A, that there is an integer $K_1 \leqq K'$, such that the quantities $c_n K_1^n$ are integers, and also from the fact that the singularities of $\theta(x)$, except for the point at infinity, lie within the circle with center at 1, and radius $1/(K_1-1)$. Accordingly, by Theorem II, as generalized above, it follows that $\theta(x)$ must be of the form $P(x)/(1-x)^h$. But this is impossible, since we assumed $\theta(x)$ to have a singularity at some point other than the center of the circle. Hence ζ must be a transcendental number.

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