A CLASS OF TRANSCENDENTAL NUMBERS

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The purpose of this short note is to determine a class of transcendental numbers by considerations relating to series of powers. I shall give here a theorem quite different from those that I have given previously.* I shall prove the following theorem.

**THEOREM I.** Let \( \sum a_n x^n = F(x) \) be a power series having singularities only within the circle with center at the point 1 and radius \( 1/(K-1) \), \( K \) an integer, there being at least one essential singularity within the circle, not at the center.

If there exists a number \( \xi \) and an integer \( K' < \xi \) such that the quantities

\[
K''\left(a_n + \frac{1}{\xi^{n+1}}\right) = N_n \quad (n = 1, 2, \ldots)
\]

are integers, then \( \xi \) is a transcendental number.

I have given elsewhere the following theorem.†

**THEOREM II.** If the series \( \sum b_n x^n = f(x) \) represents a function having singularities only within the circle with center at 1 and radius \( 1/(K_1-1) \), where \( K_1 \) is an integer, and if the quantities

\[
b_n K_1^n = N'_n
\]

are integers, then we have

\[
f(x) = \frac{P(x)}{(1 - x)^h},
\]

in which \( h \) is an integer and \( P(x) \) a polynomial.

From this theorem, I then proved the following theorem.

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* Comptes Rendus, Feb. 1, 1926; Journal de Mathématiques, 1926.
† Journal de Mathématiques, 1926.
THEOREM III. If a series \( \sum c_n x^n = \phi(x) \) has at least one singular point distinct from the point 1, and if an integer \( K_2 \) exists so that the quantities

\[
K_2^n c_n = N_n^{(2)}, \quad (n = 1, 2, \ldots)
\]

are also integers, then, if we denote by \( x_0 \) the argument of the singularity of \( \phi(x) \) which is farthest from the point 1, we shall have

\[
| x_0 - 1 | > \frac{1}{K - 1}.
\]

In these theorems, the point at infinity was counted with the other points; that is, in Theorem II, we assumed the point at infinity to be a regular point for \( \sum b_n x^n \). Likewise in Theorem III, if the point at infinity happens to be a singular point, the inequality (2) is devoid of meaning. But we can now generalize Theorem II by adding that if the point at infinity is a pole for \( f(x) \), and if the other singularities be within the circle above mentioned, we shall still have

\[
f(x) = \frac{P(x)}{(1 - x)^h},
\]

if the property (1) is satisfied.

The proof in this case will be almost the same as for II*. We will only remark that Hurwitz's theorem can be generalized to series of the form

\[
f(x) = \sum_0^\infty \frac{\alpha_n}{x^{n-m}}, \quad \phi(x) = \sum_0^\infty \frac{\beta_n}{x^{n-m}},
\]

viz., that the series

\[
\frac{A_m}{x^m} + \cdots + \frac{A}{x} + A_0 + \sum \frac{\alpha_{n+m} \beta_{m+1}}{x^n} - \frac{C'}{\alpha_{n+m-1}} \beta_{m+1} + \cdots + (-1)^n \beta_{m+n} \alpha_{n+1}
\]

* See Journal de Mathématiques, 1926.
has as singularities, besides the point at infinity, only points of the form \( \alpha + \beta \), where \( \alpha \) is a singular point of \( f(x) \) and \( \beta \) a singular point of \( \phi(x) \).

A. We can therefore see that if the series \( \sum a_n x^n \) is such that there exists an integer \( K \) so that the quantities \( a_n K^n \) are integers, then if \( P(x) \) is a polynomial with integral coefficients, the function

\[
P(x) \sum a_n x^n = \sum b_n x^n
\]

is such that there exists an integer \( K_1 \leq K \), so that the quantities \( K_1^n b_n \) are integers. This results immediately from the form of the coefficients \( b_n \):

\[
b_n = a_n A_0 + a_{n-1} A_1 + \cdots + a_{n-p} A_p,
\]

where

\[
P(x) = A_0 + A_1 x + \cdots + A_p x^p.
\]

We write then, returning to the theorem to be proved,

\[
\sum c_n x^n = \sum \left( a_n + \frac{1}{\xi^n} \right) x^n = F(x) + \frac{1}{\xi - x}.
\]

If we suppose that \( \xi \) is an algebraic number, there will be a polynomial with integral coefficients such that \( P(x)/(\xi - x) \) will be regular in the entire plane except for the point at infinity. Hence the function \( \theta(x) = P(x) \cdot F(x) \) has singularities only within the circle with center at the point 1 and radius \( 1/(K_1 - 1) \), except for a pole at infinity.

This results from the remark A, that there is an integer \( K_1 \leq K' \), such that the quantities \( c_n K^n_1 \) are integers, and also from the fact that the singularities of \( \theta(x) \), except for the point at infinity, lie within the circle with center at 1, and radius \( 1/(K_1 - 1) \). Accordingly, by Theorem II, as generalized above, it follows that \( \theta(x) \) must be of the form \( P(x)/(1-x)^h \). But this is impossible, since we assumed \( \theta(x) \) to have a singularity at some point other than the center of the circle. Hence \( \xi \) must be a transcendental number.

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