GENERALIZATION OF THE BELTRAMI EQUATIONS TO CURVED n-SPACE*

BY G. E. RAYNOR

Let $S$ be a curved $n$-space in which the linear element is given by the equation

$$ds^2 = \sum E_{ij} dx_i dx_j, \quad (i,j = 1,2, \cdots, n).$$

Without loss of generality, we may suppose

$$E_{ij} = E_{ji}.$$  \hspace{1cm} (2)

Also let $U^{(i)}$, ($i = 1, 2, \cdots, n$), be a set of $n$ independent functions of $x_1, x_2, \cdots, x_n$.

We shall say that the $U^{(i)}$ are isothermal in $S$ provided they satisfy the relation

$$\sum (dU^{(i)})^2 = \lambda \sum E_{ij} dx_i dx_j,$$  \hspace{1cm} (3)

where $\lambda$ is a function of the $x_i$ only.

If in (3) we express the $dU^{(i)}$ in terms of the differentials of $x_1, x_2, \cdots, x_n$ it follows from the independence of these differentials that the coefficients of corresponding terms on the two sides of the equation are equal and we obtain the $n(n+1)/2$ equations

$$\sum_{k=1}^{n} U^{(k)}_x U^{(k)}_y = \lambda E_{ij}. \hspace{1cm} (4)$$

Let $D$ be the discriminant of the quadratic differential form in (1) and suppose it to be written as a determinant

$$|E_{ij}|, \hspace{1cm} (5)$$

in which $E_{ij}$ is the element in the $i$th row and $j$th column. If each element of (4) be multiplied by $\lambda$ and if for $\lambda E_{ij}$ be substituted its equal given by the left side of (4), we

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readily see that the resulting determinant is the square of the Jacobian

\[ J = \frac{\partial(U^{(1)}, U^{(2)}, \ldots, U^{(n)})}{\partial(x_1, x_2, \ldots, x_n)}. \]

Hence we have

(6) \[ J = \lambda^{n/2} D^{1/2}. \]

In all that follows we shall suppose \( J \) to be written as a determinant in which \( U^{(j)}_{x_i} \), the derivative of \( U^{(j)} \) with respect to \( x_i \), is the element of \( J \) in the \( i \)th row and \( j \)th column.

Multiply both sides of (6) by \( U^{(j)}_{x_i} \), on the left letting the factor go into the \( i \)th row of \( J \). Now if we multiply each row of \( J \), other than the \( i \)th, by its \( j \)th element and add the corresponding products to the elements in the \( i \)th row, (6) becomes by means of (2) and (4)

(7) \[ \lambda J_{ij} = \lambda^{n/2} D^{1/2} U^{(j)}_{x_i}, \]

where \( J_{ij} \) is the determinant obtained from \( J \) by replacing its \( i \)th row by the \( j \)th row of \( D \). From (6) and (7) we obtain

(8) \[ U^{(j)}_{x_i} = \frac{J_{ij}}{J^{(n-2)/n} D^{i/n}}, \quad (i, j = 1, 2, \ldots, n). \]

This last set of \( n^2 \) equations are of the form obtained, by a different method, by Hedrick and Ingold* for curved 3-space. Their equation in our notation may be written

(9) \[ U^{(j)}_{x_i} = PJ_{ij}, \]

where \( P \) is an unspecified factor of proportionality. However, it may be seen as follows that equations (9) are equivalent to (8). Replace each element of \( J \) by its expression given by (9) and we obtain

(10) \[ J = P^n |J_{ij}|. \]

* Transactions of this Society, vol. 27 (1925), p. 561.
Now each element $J_{ij}$ of the determinant $|J_{ij}|$ is a determinant which has one row of $D$ in it. If we expand this determinant by cofactors with respect to the elements of this row we find that $|J_{ij}|$ breaks up into the product

$$DA_j$$

where $A_j$ is the adjoint of $J$. Hence (10) becomes

$$J = P^nDJ^{n-1},$$

and

$$P = \frac{1}{J^{(n-2)/n}D^{1/n}}.$$

If $n = 2$, equations (8) become

$$(11) \quad U_{x_i}^{(i)} = \frac{J_{ij}}{D^{1/2}}, \quad (i, j = 1, 2),$$

which are precisely the well known Beltrami equations of differential geometry. These equations have the property that the derivatives of either one of the $U^{(i)}$ are expressed in terms of the $E_{ij}$ and the derivatives of the other $U$ only. This is not the case for $n > 3$ in (8), since $J$ on the right contains $U_{x_j}$ which appears on the left. To obtain a more desirable form we proceed as follows.

From the sub-set of the equations in (8) obtained by keeping $i$ fixed, we get by taking ratios,

$$(12) \quad U_{x_k}^{(i)} = \frac{J_{ik}}{J_{ij}} U_{x_i}^{(i)}, \quad (k \neq j).$$

After substituting these expressions for $U_{x_k}^{(i)}$ in the $i$th row of $J$ on the right side of (8), $U_{x_j}^{(i)}$ can be removed as a factor from this row and solving the resulting equation for $U_{x_i}^{(i)}$ we obtain

$$(13) \quad U_{x_i}^{(i)} = \frac{J_{ij}}{M_i^{(n-2)/(2n-2)}D^{1/(2n-2)}},$$

where by $M_i$ we mean the determinant obtained from $J$ by replacing its $i$th row by the $i$th row of the determinant $|J_{ij}|$. 

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Equations (13) contain none of the derivatives of \( U^{(i)} \) on the right and hence we have all the partial derivatives of \( U^{(i)} \) expressed in terms of the \( E_{ij} \) and \( U^{(k)}_{z_j} \), \( (k \neq i) \). Hence (13) is the desired generalization of the Beltrami equations to curved \( n \)-space.

It can readily be shown, conversely, that if a set of functions satisfy (8) or (13) they also satisfy (3).*

We now proceed to find the differential equation satisfied by each of the \( U^{(i)} \) singly. Let \( C_{ij} \) denote the cofactor of the element in the \( i \)th row and \( j \)th column of \( J \). Then in (8), if we expand the \( J_{ij} \) by cofactors with respect to the row of \( E \)'s contained in them, (8) can be written

\[
\sum_{j=1}^{n} E_{kj} C_{ij} = J^{(n-2)/n}D^{1/n}U^{(i)}_{z_k}, \quad (i, k = 1, 2, \cdots, n).
\]

If out of the above set of \( n^2 \) equations we solve the set of \( n \), obtained by holding \( i \) fixed, for the \( C_{ij} \) we get

\[
C_{ij} = \frac{N_{ij}J^{(n-2)/n}}{D^{(n-1)/n}},
\]

where \( N_{ij} \) is the determinant obtained from \( D \) by substituting the \( i \)th row of \( J \) for the \( j \)th row of \( D \). If now \( J \) in the last equation be expanded with respect to cofactors of the \( i \)th row, (15) becomes

\[
C_{ij} = \frac{N_{ij}\left\{ \sum_{k=1}^{n} U_{z_k}^{(i)}C_{ib} \right\}^{(n-2)/n}}{D^{(n-1)/n}}, \quad (j = 1, 2, \cdots, n).
\]

From (16) we get, by taking ratios,

\[
C_{ik} = \frac{N_{ik}}{N_{ij}} C_{ij}, \quad (k \neq j).
\]

Substituting these values for \( C_{ik} \), in the right of (16), and, solving the resulting equation for \( C_{ij} \), we have

* See Hedrick and Ingold, loc. cit., p. 562.
Now by a well known property of Jacobians,*

$$\sum_{i=1}^{n} \frac{\partial C_{ij}}{\partial x_j} = 0. \tag{19}$$

Hence, if in (19) the expressions on the right of (18) be substituted for $C_{ij}$, we will have the differential equation satisfied by $U^{(i)}$ alone. It is readily seen that the form of this equation is independent of the index $(i)$ and hence the $n$ functions

$$U^{(1)}, U^{(2)}, \ldots, U^{(n)}$$

satisfy the same differential equation, which may be looked upon as a generalization of Laplace's equation to curved $n$-space.

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THE NON-EXISTENCE OF A CERTAIN TYPE OF REGULAR POINT SET†

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In a paper not yet published,‡ I have shown that a regular§ connected point set which consists of more than one point and remains connected upon the omission of any connected subset, is a simple closed (Jordan) curve. As a simple closed curve is a bounded point set, it is clear that there does not exist any unbounded regular connected point set which remains connected upon the omission of any connected subset.

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‡ See, however, this Bulletin, vol. 32 (1926), p. 591, paper No. 35.
§ That is, connected im kleinen.