A GENERALIZATION OF RECURRENTS

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1. Introduction. It is well known that if

$$\phi(x) = \sum_{r=0}^{\infty} \phi_r x^r, \quad \psi(x) = \sum_{s=0}^{\infty} \psi_s x^s$$

are two singly infinite series, then the coefficients in the expansion of $\phi(x)/\psi(x)$, $\log \phi(x)$, $e^{\phi(x)}$ can all be expressed as determinants in the quantities $\phi_r, \psi_s$. These expressions are called *recurrents* and have been used by several writers\(^*\) to evaluate determinants involving the binomial coefficients, Bernoulli numbers, etc.

In the present paper, the analogous results are given for the quotient of two *doubly* infinite series, and the logarithm and exponential of a doubly infinite series. The extension to $m$-tuply infinite series is briefly sketched in §8.

It is believed the expressions obtained are new; there is no reference to any such work in the four volumes of Muir's *History*. We assume throughout that all the series involved are absolutely convergent, so that the derangements and multiplications employed are justified. As a matter of fact, we are dealing essentially with infinite sets of quantities $A_{rs}$, $B_{rs}$, $C_{rs}$, $\cdots$ ($r, s = 0, 1, 2, \cdots$); the "variables" which appear in the series are merely convenient carriers for their coefficients.

We shall use, wherever convenient, the convention employed by writers on relativity for summations, namely,

$$U_{rs} x^r y^s,$$

which is taken to mean

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} U_{rs} x^r y^s,$$

the summations being understood.

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\(^*\) Muir's *History*, vols. II, III, IV, Chapters on recurrents.
2. **Degree and Rank.** Given a doubly infinite series

\[ U(x, y) = U_{rs}x^r y^s, \]

we shall invariably write \( U \) in the order

\[ U_{00} + U_{10}x + U_{01}y + U_{20}x^2 + U_{11}xy + U_{02}y^2 + \cdots, \]

or as

\[ U(x, y) = \sum_{l=0}^{\infty} \sum_{k=0}^{l} U_{l-k,k} x^{l-k} y^k. \]

We define

\[ l = (l - k) + k, \quad u_{lk} = \frac{l(l + 1) + 2(k + 1)}{2}, \]

as the **degree** and **rank** respectively of the coefficient \( U_{l-k,k} \).

Hence the degree of a coefficient is the degree of the term it multiplies. The rank of a coefficient is simply its place in the series (2). For since from (2) there are \( l+1 \) terms of degree \( l \), the coefficient \( U_{10} \) appears in the \([1 + 2 + 3 + \cdots + l] + 1\)st place, that is,

\[ u_{10} = \frac{l(l + 1)}{2} + 1. \]

The coefficient \( U_{l-k,k} \) is \( k \) terms to the right of \( U_{10} \), so that its rank is

\[ \frac{l(l + 1)}{2} + 1 + k = \frac{l(l + 1) + 2(k + 1)}{2} = u_{lk}. \]

Thus for \( U_{rs} \), the degree is \( r+s \), and the rank is

\[ u_{rs} = \frac{(r + s)(r + s + 1) + 2(s + 1)}{2}. \]

Moreover, it follows from the meaning of rank, that given any positive integer \( n \), the equation

\[ n = u_{rs} \]

determines a unique pair of non-negative integers \( r, s \), and hence a unique coefficient \( U_{rs} \) in the series (2). Let \( k \) be any integer not greater than \( r+s \). Then, by (3),
\[ u_{r+s-k,k} = \frac{(r + s)(r + s + 1) + 2(k + 1)}{2}; \]

hence
\[ u_{r+s-k,k} + s - k = \frac{(r + s)(r + s + 1) + 2(s + 1)}{2} = u_{rs}. \]

In particular
\[ u_{r+s,0} + s = u_{rs}, \quad s \leq r + s. \]

3. Coefficients for a Product. If
\[ A(x, y) = A_{qr}x^qy^r, \]
\[ B(x, y) = B_{st}x^s y^t, \]
then we know that
\[ A(x, y) \cdot B(x, y) = C_{uv}x^u y^v, \]
where
\[ C_{uv} = \sum_{\sigma=0}^{u} \sum_{\tau=0}^{v} A_{u-\sigma,v-\tau}B_{\sigma\tau}. \]

4. Coefficients for a Quotient. Consider now
\[ P(x, y) = P_{uv}x^u y^v, \]
\[ Q(x, y) = Q_{qr}x^q y^r, \]
and let
\[ \frac{P(x, y)}{Q(x, y)} = Z(x, y) = Z_{st}x^s y^t, \]
where the coefficients \( Z_{st} \) are to be determined.

First, we can assume \( Q_{00} \neq 0 \). For, if \( Q_{00} = Q_{10} = Q_{01} = \cdots = 0, Q_{ij} \neq 0 \), multiply both sides of (7) by \( x^t y^i \), replacing \( Q(x, y)/x^t y^i \) by \( Q'(x, y) \) and \( x^t y^i Z(x, y) \) by \( Z'(x, y) \) with \( Z_{00}' = Z_{10}' = Z_{01}' = \cdots = 0, Z_{ij}' = Z_{00} \). We then have a new equality of the same form as (7) with \( Q_{00}' = Q_{ij} \neq 0 \). Thus \( P(x, y) = Q(x, y)Z(x, y) \), or, by (6),
\[ P_{uv} = \sum_{\sigma=0}^{u} \sum_{\tau=0}^{v} Q_{u-\sigma,v-\tau}Z_{\sigma\tau}, \]
\[ (u,v = 0,1,2, \cdots). \]

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It may be noted in passing, that just as recurrences are related to difference equations with constant coefficients, so may (7), if \( Q(x, y) \) be a polynomial, be looked upon as a linear partial difference equation with constant coefficients to determine \( Z_{uv} \).

Let us introduce the symbol

\[
(\lambda_{r-k,k} ; u - r + k, v - k),
\]
defined by the relations

\[
\left\{ \begin{array}{l}
\lambda_{r-k,k} = \frac{r(r + 1) + 2(k + 1)}{2}, \\
(\lambda_{r-k,k} ; u - r + k, v - k) = 0, \text{ if } r > u + k, \text{ or } k > v, \\
= Q_{u-r+k,v-k}, \text{ if } k \leq v \text{ and } r \leq u + k.
\end{array} \right.
\]

Then (8) may be written

\[
(10) \sum_{r=0}^{u+v} \sum_{k=0}^{r} (\varepsilon_{r-k,k} ; u - r + k, v - k)Z_{r-k,k} = P_{uv},
\]

\[
(p_{uv} = 1, 2, 3, \ldots).
\]

For \( \lambda_{r-k,k} = \varepsilon_{r-k,k} \), by definition of rank in (3). Moreover setting \( r-k=\sigma, k=r \) in (10), we see that every term that occurs in (8) occurs in (10), and conversely.

Finally, by virtue of (9) we may replace (10) by

\[
(11) \sum_{r=0}^{n} \sum_{k=0}^{r} (\varepsilon_{r-k,k} ; u - r + k, v - k)Z_{r-k,k} = P_{uv},
\]

\[
(p_{uv} = 1, 2, 3, \ldots),
\]

where \( n \) is any integer \( \geq u+v \). Take \( n=\varepsilon_{ij} \) and consider the set of \( n=p_{ij} \) equations (11), in the \( n \) unknowns \( Z_{00}, Z_{10}, Z_{01}, \ldots, Z_{ij} \),

\[
(12) \left\{ \begin{array}{l}
Q_{00}Z_{00} = P_{00}, \\
Q_{01}Z_{00} + Q_{00}Z_{10} = P_{10}, \\
\ldots \\
Q_{ij}Z_{00} + (2 ; i - 1, j + 1)Z_{10} + \cdots + Q_{00}Z_{ij} = P_{ij}.
\end{array} \right.
\]
Since $Q_{00} \neq 0$, we have, solving for $Z_{ij}$ by determinants,

\[
\begin{vmatrix}
Q_{00} & 0 & 0 & \cdots & 0 & P_{00} \\
Q_{10} & Q_{00} & 0 & \cdots & 0 & P_{10} \\
Q_{01} & 0 & Q_{00} & \cdots & 0 & P_{01} \\
Q_{20} & Q_{10} & 0 & \cdots & 0 & P_{20} \\
Q_{11} & Q_{01} & Q_{10} & \cdots & 0 & P_{11} \\
Q_{02} & 0 & Q_{01} & \cdots & 0 & P_{02} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
Q_{ij} & (2; i - 1, j + 1), (3, i - 2, j + 2) & \cdots & \cdots & \cdots & \cdots & \cdots & P_{ij}
\end{vmatrix}
\]

where (13) is constructed on the following scheme. The elements in the $s$th column ($s < n$) consist of $s - 1$ zeros, then the coefficients of $Q(x, y)$ of degree zero, one, two, three, \ldots, in their proper order, the groups of coefficients of the same degree being separated by $r'$ zeros, where $s = z_{r' - k', k'}$ determines $r'$. In fact, we see from (11), that the elements in the $s$th column are given by the expression

\[(14) \quad (s ; u - r' + k', v - k'),\]

where $r'$ and $k'$ are determined from the equation

\[z_{r' - k', k'} = s,
\]

in accordance with (4), and $(u, v)$ has the successive values

\[(15) \quad (0, 0), (1, 0), (0, 1), (2, 0), (1, 1), \ldots, (u, v), \ldots, (i, j),
\]

$n = \lambda_{ij}$ in number, $(u, v)$ appearing in the $\lambda_{uv}$th place in (15). Now the $\sigma + 1$ terms $(uv)$ of constant degree $u + v = \sigma$, $(0 \leq \sigma \leq i + j)$, appear in the order

\[(16) \quad (\sigma, 0), (\sigma - 1, 1), \ldots, (\sigma - k, k), \ldots, (0, \sigma).
\]

If $n$ is replaced $u$ by $\sigma - v$, (14) becomes

\[(s ; \sigma - r' + k' - v, v - k'),\]
so that for \( \nu = 0, 1, 2, \ldots, \sigma \), we have the values of (14) for the sequence (16). But this expression vanishes, by (9), unless

(a) \( v - k' \geq 0 \),
(b) \( \sigma - r' + k' - v \geq 0 \).

Thus the first \( k' \) terms of (16) yield \( k' \) zeros. Replace \( v \) by \( r + k' \), where to satisfy (a) and (b) \( 0 \leq r \leq \sigma - r' \geq 0 \).

We thus obtain from the next \( \sigma - r' + 1 \) terms of (15),

\[
(s ; \sigma - r', 0), (s ; \sigma - r' - 1, 1), \ldots, (s ; 0, \sigma - r'),
\]

or by (9),

\[
Q_{\sigma - r', 0}, Q_{\sigma - r' - 1, 1}, \ldots, Q_{0, \sigma - r'},
\]

the coefficients of \( Q(x, y) \) of degree \( \sigma - r' \) in their proper order. The remaining terms of (16) produce \( \sigma + 1 - k' - (\sigma - r' + 1) = r' - k' \) zeros.

Since the sequence of degree \( \sigma + 1 \) following (16) produces \( k' \) zeros followed by

\[
Q_{\sigma + 1 - r', 0}, Q_{\sigma + 1 - r' - 1, 1}, \ldots, Q_{0, \sigma + 1 - r'},
\]

we see that the coefficients of degree \( \sigma - r' = 0, 1, 2, 3, \ldots \) are separated by \( r' \) zeros, as stated. \( Q_{00} \) appears when \( \sigma - r' = 0 \). From (a) and (b),

\[
v = k', \quad u = \sigma - v = r' - k'.
\]

Hence \( Q_{00} \) appears in the \( \lambda_{r' - k'}, k' \) or the \( s \)th place in the column; i. e., in (13), the elements \( Q_{00} \) lie along the main diagonal. The last column in (13) consists of the elements \( P_{00}, P_{01}, \ldots, P_{ij} \) in order of rank.

5. Final Coefficients in the \( s \)th Column. There is some doubt about the last few elements in the \( s \)th column, but this is obviated as follows. Take \( s = z_{i + j - r, r}, (0 \leq r \leq j - 1) \), i. e., consider the \((n - j)\)th, \((n - j - 1)\)th, \ldots, \((n - 1)\)th columns of (13).

We have then \( s = n - j + \tau \) so that the \( s \)th column contains \( n - j + \tau \) zeros, \( Q_{00} \), followed by \( i + j \) zeros by our results in §4. But since there are only \( n \) elements in the column, \( Q_{00} \) is followed by \( j - \tau - 1 \) zeros, since \( j - \tau - 1 \) is always
less than $i+j$. Hence we can reduce (13) to a determinant of the $(n-j)$th order multiplied by $Q_0^j$ to some power, for the $j$ columns just considered consist entirely of zeros save along the main diagonal where $Q_0$ appears.

Now $n = z_{ij}$ and $z_{i+j,0+j} = n$, by (5). Hence, setting $n-j = \nu$, we have

$$n - j = z_{i+j,0} = \frac{(i+j)(i+j+1)}{2} + 1 = \nu.$$ 

The elements in the $\nu$th row are now

$$Q_{ij}, (2 ; i - 1, j + 1), (3 ; i - 2, j + 2), \ldots ,$$

$$(\nu - 1 ; i - \nu + 2, j + \nu - 2), P_{ij}$$

so that the $s$th column terminates with

$$(s ; i - s + 1, j + s - 1)$$

and in the $(\nu-1)$th row the elements are

$$(z_{r-k,k} ; k - r, i + j - 1 - k) = \delta_{r+1,k+i} Q_{0,i+j-1-r},$$

$$(r,k = 0, 1, 2, \ldots , i + j - 1),$$

where $\delta_{uv}$ is the Kronecker symbol.

Thus we have

$$Q_{0,i+j-1,0}, Q_{0,i+j-2,0}, 0, 0, Q_{0,i+j-3,0}, 0, 0, \ldots , Q_{00}, P_{0,i+j-1},$$

so that our evaluation of $Z_{ij}$ gives us

$$Q_{00} Z_{ij} =$$

$$|$$

$$Q_{00} 0 \ldots \ldots P_{00}$$

$$Q_{10} Q_{00} \ldots \ldots P_{01}$$

$$Q_{01} 0 \ldots \ldots P_{10}$$

$$Q_{20} Q_{10} \ldots \ldots P_{20}$$

$$Q_{11} Q_{01} \ldots \ldots P_{11}$$

$$Q_{02} 0 \ldots \ldots P_{02}$$

$$Q_{ij} (2 ; i - 1, j + 1), \ldots \ldots P_{ij}$$

$$|$$

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where
\[ \nu = \frac{(i + j)(i + j + 1)}{2} + 1. \]

6. Expression of the Z's as Recurrents. There remains still one more simplification; the quantities \( Z_{i+j,0}, Z_{0,i+j} \) can be expressed as recurrents. For we obtain from (7), §4, by the ordinary multiplication rule

\[ P_{uv} = \sum_{t=0}^{u} \sum_{s=0}^{v} Q_{ts} Z_{u-t, v-s}, \quad (u, v = 0, 1, 2, 3, \ldots). \]

This result may be written

\[ P_{uv} - R_{uv} = \sum_{t=0}^{u} \sum_{s=0}^{v} Q_{ts} Z_{u-t, v-s}, \quad (u, v = 0, 1, 2, 3, \ldots), \]

where

\[ R_{uv} = \sum_{t=0}^{u} \sum_{s=0}^{v} Q_{ts} Z_{u-t, v-s}, \]

so that

\[ R_{u0} = 0, \]
\[ R_{u1} = \sum_{t=0}^{u} Q_{t1} Z_{u-t, 0}, \]
\[ R_{u2} = \sum_{t=0}^{u} Q_{t1} Z_{u-t, 1} + \sum_{t=0}^{u} Q_{t2} Z_{u-t, 0}, \]

etc.

The formula (19) gives for \( u = 0, 1, 2, 3, \ldots, i+j \), the set of \( i+j+1 \) equations

\[ Q_{0v} Z_{0v} = P_{0v} - R_{0v}, \]
\[ Q_{10} Z_{0v} + Q_{00} Z_{1v} = P_{1v} - R_{1v}, \]
\[ Q_{20} Z_{0v} + Q_{10} Z_{1v} + Q_{00} Z_{2v} = P_{2v} - R_{2v}, \]
\[ \vdots \]
\[ Q_{i+j,0} Z_{0v} + Q_{i+j-1,0} Z_{1v} + Q_{i+j-2,0} Z_{2v} + \cdots + Q_{00} Z_{i+j,v} = P_{i+j,v} - R_{i+j,v}, \]

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so that

\[ Q_{i+j+1}^{i+j+1}Z_{i+j,v} = \begin{vmatrix} Q_{00} & 0 & \cdots & P_{0v} - R_{0v} \\ Q_{10} & Q_{00} & \cdots & P_{1v} - R_{1v} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{i+j,0} & \cdots & P_{i+j,v} - R_{i+j,v} \end{vmatrix}. \]

In particular

\[ Q_{00}^{i+j+1}Z_{i+j,0} = \begin{vmatrix} Q_{00} & 0 & \cdots & P_{00} \\ Q_{10} & Q_{00} & \cdots & P_{10} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{i+j,0} & \cdots & P_{i+j,0} \end{vmatrix}. \]

which gives the required expression for \( Z_{i+j,0} \) as a recurrent. From symmetry the expression for \( Z_{0,i+j} \) is derived from (22) by simply interchanging the subscripts of all the terms in (21).

We may observe that, having obtained the quantities \( Z_{u0} \) by (22), we know \( R_{u1} \), so that we can calculate the quantities \( Z_{i+j-1,1} \) by means of (21). Proceeding thus step by step, we can finally calculate \( Z_{ij} \).

There are a number of relations among the determinants (17), (22). For example, suppose we interchange \( x \) and \( y \) in the equation (7),

\[ \frac{P(x,y)}{Q(x,y)} = Z(x,y). \]

The effect is merely to interchange the subscripts of the coefficients throughout. Hence in (17), we can interchange the subscripts of \( Z_{ij} \), the \( Q \)'s and the \( P \)'s and obtain an expression for \( Z_{ji} \), \( \nu \), the order of the determinant, being unaffected by the process. Again, we may write (7) as

\[ \frac{P(x,y)}{Z(x,y)} = Q(x,y), \]

so that we can interchange the roles of the \( Q \)'s and \( Z \)'s in (17).
7. Final Expressions for the Z's. The first eleven coefficients in the development of

\[
\frac{P(x, y)}{Q(x, y)} = Z_{00} + Z_{10}x + Z_{01}y + \cdots + Z_{03}y^3 + \cdots
\]

are

\[
Z_{00} = Q_{00}^{-1}P_{00},
\]

\[
Z_{10} = Q_{00}^{-2} \begin{vmatrix} Q_{00} & P_{00} \\ Q_{10} & P_{10} \end{vmatrix}, \quad Z_{10} = Q_{00}^{-3} \begin{vmatrix} Q_{00} & P_{00} \\ Q_{01} & P_{01} \end{vmatrix},
\]

\[
Z_{20} = Q_{00}^{-3} \begin{vmatrix} Q_{00} & 0 & P_{00} \\ Q_{10} & Q_{00} & P_{10} \\ Q_{20} & Q_{10} & P_{20} \end{vmatrix}, \quad Z_{02} = Q_{00}^{-3} \begin{vmatrix} Q_{00} & 0 & P_{00} \\ Q_{01} & Q_{00} & P_{01} \\ Q_{02} & Q_{01} & P_{02} \end{vmatrix},
\]

\[
Z_{11} = Q_{00}^{-4} \begin{vmatrix} Q_{00} & 0 & 0 & P_{00} \\ Q_{10} & Q_{00} & 0 & P_{10} \\ Q_{01} & 0 & Q_{00} & P_{01} \\ Q_{11} & Q_{01} & Q_{10} & P_{11} \end{vmatrix},
\]

\[
= Q_{00}^{-4} \begin{vmatrix} Q_{00} & 0 & 0 & P_{00} \\ Q_{01} & Q_{00} & 0 & P_{01} \\ Q_{10} & 0 & Q_{00} & P_{10} \\ Q_{11} & Q_{10} & Q_{01} & P_{11} \end{vmatrix},
\]

\[
Z_{30} = Q_{00}^{-4} \begin{vmatrix} Q_{00} & 0 & 0 & P_{00} \\ Q_{10} & Q_{00} & 0 & P_{10} \\ Q_{20} & Q_{10} & Q_{00} & P_{20} \\ Q_{30} & Q_{20} & Q_{10} & P_{30} \end{vmatrix},
\]

\[
Z_{03} = Q_{00}^{-4} \begin{vmatrix} Q_{00} & 0 & 0 & P_{00} \\ Q_{01} & Q_{00} & 0 & P_{01} \\ Q_{02} & Q_{01} & Q_{00} & P_{02} \\ Q_{03} & Q_{02} & Q_{01} & P_{03} \end{vmatrix},
\]
8. Quotient of two m-tuply Infinite Series. The same method can be applied to the development of the quotient of two triply, or indeed of two m-tuply infinite series. We need only to generalize the formulas for degree and rank of §2, for product of two series in §3 and to introduce a symbol corresponding to the \( \lambda_{r-k, h; u-r+k, v-k} \) of §4 to obtain the analog of (17); the analog of (21) is obtained with equal ease. Thus for the m-tuply infinite series,

\[
A(x_1, \ldots, x_m) = A_{i_1, i_2, \ldots, i_m} x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m},
\]

\( i_1 + i_2 + \cdots + i_m \) is the degree of the coefficient \( A_i \) above and

\[
(23) \quad a_{i_1, i_2, \ldots, i_m} = \sum_{r=1}^{m} \left( \begin{array}{c}
    i_r + i_{r+1} + \cdots + i_m + m - r \\
    m - r + 1
  \end{array} \right) + 1
\]

is its rank, when \( A \) is written so that the terms of degree 0, 1, 2, \ldots, \( r, r+1, \cdots \) succeed each other in order, and the terms of degree \( r \) are arranged in alphabetic order.
For the product of two such series, we have the formula
\[ A(x_1, \ldots, x_m) \cdot B(x_1, \ldots, x_m) = C(x_1, \ldots, x_m), \]
where
\[ (24) \quad C_{i_1, \ldots, i_m} = \sum_{r=0}^{j} A_{i_r, i_r, \ldots, i_r} B_{r_r, r_r, \ldots, r_r}. \]
If we define \( Z(x_1, \ldots, x_m) \) by
\[ (25) \quad \frac{P(x_1, \ldots, x_m)}{Q(x_1, \ldots, x_m)} = Z(x_1, \ldots, x_m), \]
then our new symbol is
\[ \Delta_{s} = \left( \lambda_{s}; \ j_1 - s_1 + s_2, j_2 - s_2 + s_3, \ldots, \ j_m - s_m + s_m \right), \]
defined by \( \Delta_{s} = 0 \) if \( j_r - s_r + s_{r+1} \) is negative for any \( r \) between 0 and \( m+1 \), and
\[ (26) \quad \Delta_{s} = Q_{j_m - s_m + s_m}, \]
if \( j_r - s_r + s_{r+1} \) is positive for every \( r \) between 0 and \( m+1 \), and by convention \( s_{m+1} = 0 \). But our final results in the general case are completely obscured by the symbolism introduced to express them.

9. Expansion of a Logarithm. We can readily obtain the expansion of
\[ (27) \quad \log Q(x, y) = Z(x, y) \]
where
\[ Q(x, y) = Q_{00} x^{s_0} y^{r_0}, \quad Q_{00} \neq 0, \]
\[ Z(x, y) = Z_{s1} x^{s_1} y^{r_1}. \]

For, operating on (27) with
\[ \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \]
we obtain a result of the form

\[
\frac{P(x,y)}{Q(x,y)} = W(x,y),
\]

where

\[
W_{st} = (s + t)Z_{st}, \quad P_{uv} = (u + v)Q_{uv},
\]

by Euler's theorem on homogeneous functions and our convention as to the order of an infinite series.

Thus \( Z_{00} = \log Q_{00}; \) the other coefficients are derived from our previous expressions by replacing \( Z_{ij} \) by \( Z_{ij}/(i+j) \) and \( P_{uv} \) by \( (u+v)Q_{uv}. \)

10. Expansion of an Exponential. For \( e^{Q(x,y)} \), a slightly different procedure is necessary. Let

\[
\begin{cases}
Q(x,y) = W(x,y), \\
\partial = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}
\end{cases}
\]

We shall have

\[
\begin{cases}
Q(x,y) = Q_{or}x^ay^r, \quad \theta Q = (q + r)Q_{or}x^ay^r, \\
W(x,y) = W_{st}x^sy^t, \quad \theta W = (s + t)W_{st}x^sy^t.
\end{cases}
\]

Then

\[
\theta e^{Q} = \theta Q e^{Q} = (\theta Q) \cdot W = \theta W.
\]

Hence, by (6),

\[
(u + v)W_{uv} = \sum_{\sigma=0}^{u} \sum_{\tau=0}^{v} (u + v - \sigma - \tau)Q_{u-\sigma,v-\tau}W_{\sigma\tau}.
\]

Now as in §4 introduce a symbol

\[
(\lambda_{r-k,k}; u - r + k, v - k)
\]

defined as in (9), with one modification, namely, while

\[
\lambda_{r-k,k} = \frac{r(r + 1) + 2(k + 1)}{2}
\]
and

\[(\lambda_{r-k,k} ; u - r + k, v - k) = 0, \text{ if } r > u + k, \text{ or } k > v, \]

\[= (u + v - r)Q_{u-r+k,v-k}, \]

if \( k \leq v \) and \( r \leq u + k, \)

we have

\[(\lambda_{r-k,k} ; u - r + k, v - k) = -(u + v), \]

for \( k = v \) and \( r = u + v. \) Then (31) may be written in the form

\[\sum_{r=0}^{u+v} \sum_{k=0}^{r} (W_{r-k,k} ; u - r + k, v - k)W_{r-k,k} = 0, \]

just as (10) was equivalent to (8) in \$4. Also \((\lambda_{00} ; 0, 0) = 0, \)

but from (28) we see that

\[e_{00} = W_{00} = -P_{00}, \text{ say.} \]

Thus (28) becomes equivalent to (10) if we replace in each equation \( P_{uv} \) by 0 for \( u+v > 0 \) and \( (\lambda_{uv} ; 0, 0) \) by \(-(u+v), \)

instead of by \( Q_{00}. \) We thus obtain the following set of \( w_{ij} \)

\[\text{equations for } W_{ij}: \]

\[- W_{00} = P_{00}, \]

\[1 \cdot Q_{10}W_{00} - 1 \cdot W_{10} = 0, \]

\[1 \cdot Q_{01}W_{00} + 0 \cdot W_{10} - 1 \cdot W_{01} = 0, \]

\[2 \cdot Q_{20}W_{00} + 1 \cdot Q_{10}W_{10} + 0 \cdot W_{01} - 2W_{20} = 0, \]

\[2 \cdot Q_{11}W_{00} + 1 \cdot Q_{01}W_{10} + 1 \cdot Q_{10}W_{01} + 0 \cdot W_{20} - 2W_{11} = 0, \]

\[2 \cdot Q_{02}W_{00} + 0 \cdot W_{10} + 1 \cdot Q_{01}W_{01} + 0 \cdot W_{20} + 0 \cdot W_{11} - 2W_{02} = 0, \]

\[\ldots \ldots \ldots \ldots \ldots \]

\[(i + j)Q_{ij}W_{00} + \cdots = -(i + j)W_{ij} = 0. \]
The determinant of this system is
\[
( - 1)( - 1)^2( - 2)^3( - 3)^4 \cdots \\
( - i - j + 1)^{i+j}( - i - j) \\
= ( - 1)^{i+j} 1^2 \cdot 2^3 \cdot 3^4 \cdots (i + j - 1)^{i+j}(i + j)^j .
\]

Just as before, if we solve for \( W_{ij} \), we can reduce the determinant we obtain corresponding to (12) to one of the \( \nu \)th order,
\[
\nu = \frac{(i + j)(i + j + 1)}{2} + 1.
\]

But we can also develop this expression with respect to its last row which is \( P_{00}, 0, 0, \cdots, 0 \), obtaining a determinant of the \((\nu - 1)\)st order with a factor \((-1)^{\nu-1}\). Hence our final form for \( W_{ij} \) is
\[
1^2 \cdot 2^3 \cdot 3^4 \cdots (i + j - 1)^{i+j} W_{ij}
\]
\[
\begin{array}{cccccc}
Q_{10} & -1 & 0 & 0 & 0 & \cdots & 0 \\
Q_{01} & 0 & -1 & 0 & 0 & \cdots & 0 \\
2Q_{20} & Q_{10} & 0 & -2 & 0 & \cdots & 0 \\
2Q_{11} & Q_{01} & Q_{10} & 0 & -2 & \cdots & 0 \\
2Q_{22} & 0 & Q_{01} & 0 & 0 & \cdots & 0 \\
& \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
& & & & & \ddots & 0 \\
(i + j) Q_{ij} & (2, i - 1, j + 1) & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
\]

As before, we can interchange subscripts of all the \( Q \)'s to obtain \( W_{i\nu} \). We can also express \( W_{i+j,0} \) and \( W_{0,i+j} \) as recurrernts; thus
(i + j - 1)!W_{i+j,0} = - \begin{vmatrix}
Q_{10} & -1 & 0 \\
2Q_{20} & Q_{10} & -2 & 0 \\
3Q_{30} & 2Q_{20} & Q_{10} & -3 & 0 \\
\vdots & \vdots & \vdots \\
(i + j)Q_{i+j,0} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{vmatrix}

with a similar expression for $W_{0,i+j}$.

11. Conclusion. It hardly seems necessary to give numerical examples of these expansions. As in the case of recurrences, from expressions of such generality any desired example may be derived by a mere substitution of numbers for letters in the general formulas. The quotient of two polynomials, the reciprocal of a series or a polynomial, for example, are included as special cases.

It appears from the expressions for $Z_{11}$ and $Z_{12}$ in §7, that a further immediate reduction of the order of the determinants (17) is sometimes possible; but to explicate this reduction in the general case would be to mar the simplicity and symmetry of our developments.

In conclusion I should like to thank Professor E. T. Bell for criticism and suggestions in the writing of this paper.

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A CORRECTION

In the paper by H. W. March, *The Heaviside operational calculus*, this Bulletin, vol. 33(1927), on page 312, in the line following equation (2), change "negative" to "positive."