ON INDUCTIVE RELATIONS*

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1. *Introduction.* Difficulties, connected with the occurrence of reflexive fallacies, appear when we attempt to give straightforward definitions of inductive series. These difficulties arise inevitably from the fact that, if a series is to be inductive, in the ordinary sense, properties of all orders must be transmitted in the series, and it would seem that no proposition, nor any finite set of propositions, can assert that properties of all orders are transmitted.† In the discussion which follows, we shall confine attention to the definition of a particular type of inductive series, namely the order-type ω. This restriction is made for the sake of definiteness and simplicity, and it does not entail any loss of generality in the points we shall wish to illustrate.

In a note in Mind for 1923, Dr. J. E. McTaggart holds that there is a sense in which the dictum, *No proposition can be about itself*, is false;‡ and he maintains, on the contrary, that propositions in intension can be about themselves in the sense that they are applicable to themselves; and that distinctions of type are transcended in the case of these propositions. I think it just possible that this distinction is an important one, and wish to inquire what can be made of it by way of overcoming certain difficulties in the theory of types. McTaggart's view requires that we make a sharp distinction between modal and non-modal propositions, and that non-modal or material propositions be strictly subject to differences of type, in contrast to modal propositions.

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† Compare, however, *Principia Mathematica*, 2d ed., vol. 1, p. xliii ff. and Appendix B.

‡ *Propositions applicable to themselves*, Mind, vol. 32, p. 462. We may, for the moment, overlook the fact that this dictum itself seems to be a proposition about itself; it is a proposition in intension. Compare L. Wittgenstein, *Tractatus Logico-Philosophicus*, p. 57.

We may begin by distinguishing, roughly, three senses in which a proposition may be said to be about its subject-matter. (1) Primary propositions, propositions like "a gives b to c," are, in a well-understood sense, directly about the terms involved in them, and express a relation among these terms, or ascribe a characteristic to a single term. (2a) Elementary functions of primary propositions, propositions of the form $p \vee q$ for example, are not directly about the terms involved in p, q; but they are about these terms indirectly, since they restrict the truth-values of p, q,—"*at least one of the propositions* p, q *is true*." Tertiary propositions, like $(p \vee q) . (r \vee s)$, are about the terms in the primary propositions still less directly; and so on. (2b) General propositions, propositions of the form $(x) . \phi x$ for example, differ from primary and elementary propositions in that proper names are replaced by apparent variables. Propositions of the form $(x).\phi x$ are not directly about the values of x; such a proposition may be rendered, Every value of ϕx is true. Propositions involving a double quantification, as for example $(\exists y):(x).\phi(x,y)$, are about the values of y indirectly, and about the values of x doubly indirectly,—some value of $(x).\phi(x,y)$, say $(x).\phi(x,b)$, is such that $(x).\phi(x,b)$ is true; and so on. Again, $\phi(x,y)$ may have elementary functions of primary propositions for values; in which case we get a combination of (2a) and (2b). Or, we may have elementary functions of functions which are not elementary, as for example $(x).\phi x. \vee .(\exists y).\psi y$. These functions are always reducible to functions which do not involve elementary functions of functions that are not elementary—the function just cited is equivalent to $(x):(\exists y).\phi x \vee \psi y$—but even when such reduction is not effected, the principle of interpretation is the same: we have a definite hierarchy of values and values of values. (3) This is the sense in which a modal proposition is about the propositions to which it applies; the sense in which, for example, "*being a proposition entails being true or false*" is applicable to a given proposition. I shall wish to express these propositions in a somewhat different form, and will defer discussion of (3) until some other distinctions have been drawn. The contrast to be emphasized here falls between (2b) and (3), between general propositions in extension and modal propositions.

We have now to consider two senses in which an expression may be said to be a value of another expression, senses in which one expression may be validly obtained from another by substitution. (i) In a strict sense, when a proposition is a value of a function, the function characterizes the proposition; the proposition is an instance of the function. Thus, $(x).\phi x$, Every value of ϕx is true, involves the totality of instances of ϕx; and it is in connection with this sense of value that illegitimate totalities arise, when we attempt to include as values propositions which are not instances of ϕx. The same remarks apply to multiply-quantified propositions—as for example, $(x):(\exists y).\phi(x,y)$, Every value of $(\exists y).\phi(x,y)$ is true—which involve a hierarchy of values. In this sense of value, ϕa is inferred from $(x).\phi x$ by substituting a for x in ϕx; and it is essential, for substitution of this sort to be valid, that a replace x in all of the occurrences of x. This point is of some importance in connection with the occurrence of certain pseudo-propositions. For example, if we should attempt to substitute ϕx for x in ϕx, we should get $\phi(\phi x)$; but this substitution is not complete, for ϕx must be substituted for the occurrence of x in this latter expression, and so on. The regression is infinite and vicious. (ii) We have, in this case, to consider a function f' as a value of a function f in the sense that the intension of f' involves that of f, and is, in general, a further determination of the intension of f. This relation holds when f' entails f. We may begin with an illustration. If we turn to the theory of elementary functions of propositions, as given in *Principia Mathematica*, vol. 1, *1–*5, and consider any particular proposition, as for example

(1) $$p.p \supset q. \supset .q,$$

there appears to be some doubt as to the precise force of this proposition; and there are, in fact, two alternative readings, either of which might be meant. It might be meant that

(2) $$(p, q) : p . p \supset q . \supset . q,$$

every value of $p . p \supset q . \supset . q$ is true. On the other hand, it might be held that (1) is a modal proposition, involving the function $p . p \supset q . \supset . q$, which could be written

(3) "$p . p \supset q . \supset . q$" cannot fail.

It is to be noted that the terms *hold* and *fail* apply to properties or functions, while *true* and *false* apply to propositions; so that (2) is about the totality of values of the function involved, while (3) asserts a modal characteristic of the function itself. Now any function which is a further determination of the function in (3), as for example

(4) $$\sim p . \sim p \supset q . \supset . q,$$

is, in a sense, a value of the function in (3). But (4) is not a value in a sense (i); for substitution of $\sim p$ for p, in sense (i), would lead to a vicious regression, since $\sim p$ would have to replace p in every occurrence of p; and in any case, the expression obtained by substitution is not a proposition, as it always is in sense (i). In the language of Mr. W. E. Johnson, (4) is a relative determinate under the function in (3) as determinable;* the function in (3) is related to (4) as "being colored" is related to "being red," and not as "x is colored" is related to "*this* is colored"—this latter being the relation of function to value in sense (i).

In the second edition of *Principia Mathematica*, the ambiguity of (1), as between (2) and (3), is decided in favor of (2).† This is natural in view of the abandonment of the real variable, as distinguished from a universal apparent variable having the whole of the asserted proposition or scope, and in view of the fact that the reference of the

* *Logic*, Part 1, p. 173.

† Introduction, vol. 1, p. xiii.

variables in (1) was confined to propositions of one order. We shall, on the other hand, adopt (3) as the interpretation of (1); and this is why numbers *1–*5 were referred to as the theory of elementary functions of propositions, rather than as the theory of elementary propositions. If we denote by p_1 a proposition of the first order, and by p_2 a proposition of the second order, then it is to be held that

$$(p_1, q_1) : p_1 . p_1 \supset q_1 . \supset q_1 \tag{2_1}$$

follows from (3), but not conversely; and that

$$(p_2, q_2) : p_2 . p_2 \supset q_2 . \supset . q_2 \tag{2_2}$$

follows from (3), but not conversely. Again, the proposition

$$(\phi) :. (\exists y) : (x) . \phi(x, y) : \supset : (x) : (\exists y) . \phi(x, y) \tag{5}$$

follows from the modal proposition

$$\text{“}(\exists y) : (x) . \phi(x, y) : \supset : (x) : (\exists y) . \phi(x, y)\text{” cannot fail ;} \tag{6}$$

and this holds no matter what the order of ϕ, in 5, may be. A modal proposition entails a corresponding factual universal, and it may entail many such propositions of different orders, but no factual universal entails a modal proposition.*

2. *Modal, Material, and Apparent Variables.* In distinguishing modal and material propositions, we may recognize three kinds of variables. The variables p, q in (3) and the variable ϕ in (6) are modal variables; p, q in (2) and x, y in (5) and (6) are, of course, apparent variables. Material variables occur only in properties or functions. Thus, in the function $(x):(\exists y).\phi(x, y)$, ϕ is a material variable, and must be of a definite order. In a function like "x is colored," x is a material variable; and we may, when necessary, indicate material variables, such as ϕ and x, by writing $\ddot{\phi}$ and $\ddot{x}$. Similarly, modal variables may be written $\bar{\phi}$, $\bar{x}$; ϕ, $\bar{x}$; so that (3) becomes $\bar{p}.\bar{p} \supset \bar{q}. \supset .\bar{q}$. If this analysis should

* See C. I. Lewis, *A Survey of Symbolic Logic*, Chap. V, on strict relations.

be right, it would seem that what originally gave rise to the notion of a real variable was the recognition of propositions in which what we have called modal variables occur; but that the real variable was confused, on the one hand, with the universal apparent variable, and on the other with modal variables.

3. *Modal and Material Properties.* A property or function is derived from a proposition by replacing constant constituents of the proposition by appropriate material variables; and this holds for both material and modal propositions, for the proposition is, in either case, a value in sense (i) of the derived function. Consider, for example, the material function

$$\phi x \supset \phi y. \tag{7}$$

This function will be satisfied by any value of x which lacks ϕ, for all values of y, and by any value of y which has ϕ, for all values of x. But the modal property,

$$\text{“}\phi x \supset \phi y\text{” cannot fail,} \tag{8}$$

will hold only for $x=y$, or when ϕ is some special function and x,y are so chosen that x having ϕ necessitates y having ϕ. The relation of (7) to (8) is simply that the values for which (8) holds are a proper part of the values for which (7) holds. Again, the relation of identity is not properly analyzed by the material function $(\phi).\phi x \supset \phi y$, but rather by the modal function $\phi x \supset \phi y$, that is, being a function of x necessitates being a function of y. Modal variables occur in modal propositions and in modal properties, and the occurrence of at least one modal variable is necessary for these propositions and properties; apparent variables occur in modal and material propositions and in modal and material properties; material variables occur in modal and material properties, and their presence is necessary and sufficient for a property.

4. *Definition of the Order-Type* ω. If we consider a series of the form $a_0, \cdots, a_n, \cdots$, which is inductive, what we mean by saying that the series is inductive is always strictly equivalent to what we mean by saying that, beginning with a_0, any term, a_n, can be reached step by step. We require an analysis of what we mean when we say that a_n is an inductive distance from a_0, or that a_n can be reached step by step from a_0; and it is clear that, if a proposition p_1 is to be a right analysis of a proposition p_2, then p_1 must entail and follow from p_2, that is, must be strictly equivalent to p_2. If now we consider some property f, which belongs to a_0, and is transmitted in the series, f will also belong to a_n; and we may say that it is not the case that $f(a_0)$ is true and $f(a_n)$ false. But, of course, if b_n be some term not belonging to the series, it may also be true that $f(b_n)$; so that we may say that it is not the case that $f(a_0)$ is true and $f(b_n)$ false. There is, however, this important difference: f belongs to a_n for the reason that is belongs to a_0 and is transmitted; whereas f does not belong to b_n for this reason.* Expressed otherwise, being an hereditary property belonging to a_0 entails being a property belonging to a_n; it follows from $f(a_0)$ and f is hereditary, alone, that $f(a_n)$, but not that $f(b_n)$. It might be supposed that we could express the fact that a_n is an inductive distance from a_0 by saying that a_n has every hereditary property which belongs to a_0; but this is not the case. For example, in a series of the form $a_0, \cdots, a_n, \cdots, a_\omega$, where a_ω is the limit of the progression, it might be true that a_ω has every hereditary property which belongs to a_0; whereas it is of course false that a_ω can be reached step by step from a_0, and false that $f(a_0)$ and f is hereditary entails $f(a_\omega)$.

Properties of the order-type ω will be formulated, in the usual way, on ϕx, $\psi(x, y)$, that is, on a class K and a dyadic relation R_2. There will be the usual properties for a discrete series having a first but no last element, which are material properties; and a single modal property. This modal prop-

* Compare Lewis, loc. cit., on strict implication.

erty distinguishes ω from such order-types as $\omega+{}^*\omega+\omega$, which also satisfy all of the material properties. A property is hereditary in $a_0, \cdots, a_n, \cdots$, if there is no term having this property whose immediate successor lacks it. Now t_2 immediately succeeds t_1 if, and only if,

$$\phi(t_1).\phi(t_2).\psi(t_1,t_2):.(z):\phi(z).z\neq t_2.\psi(t_1,z).\supset.\psi(t_2,z).$$

This is a function of t_1, t_2, and we may denote it by $S(t_1, t_2)$. A property f is hereditary in the series if, and only if,

$$(t_1,t_2):S(t_1,t_2).f(t_1).\supset.f(t_2)\ ;$$

and this function of f may be denoted by $T(f)$. If a_i, a_j are such that $\psi(a_i, a_j)$, then $T(f).f(a_i).\supset.f(a_j)$. But this relation is stronger, for $T(f).f(a_i)$ entails $f(a_j)$, that is,

$$T(\bar{f}).\bar{f}(a_i).\supset.\bar{f}(a_j)\ ;$$

so that the inductive property may be formulated,

p_1 $\qquad (a_i,a_j):.\phi(a_i).\phi(a_j).\psi(a_i,a_j).\supset:T(\bar{f}).\bar{f}(a_i).\supset.\bar{f}(a_j).$

The following material properties are familiar:

p_2 $\qquad (\exists x).\phi x;$ $\qquad\qquad p_3$ $\qquad (x).\phi x\supset\sim\psi(x,x);$

p_4 $\qquad (x_1,x_2):\phi(x_1).\phi(x_2).\sim x_1=x_2.\supset.\psi(x_1,x_2)\vee\psi(x_2,x_1);$

p_5 $\qquad (x_1,x_2):\phi(x_1).\phi(x_2).\sim x_1=x_2.\supset.\sim\psi(x_1,x_2)\vee\sim\psi(x_2,x_1);$

p_6 $\qquad (x_1,x_2,x_3):\phi(x_1).\phi(x_2).\phi(x_3).\sim x_1=x_2.\sim x_2=x_3.$
$$\sim x_1=x_3.\psi(x_1,x_2).\psi(x_2,x_3).\supset.\psi(x_1,x_3);$$

p_7 $\qquad (\exists x):.(y):\phi x:\phi y.\sim x=y.\supset.\psi(x,y);$

p_8 $\qquad (x):::(\exists y)::.(z)::\phi x.\supset:.\phi y.\psi(x,y):.\phi z.\supset:$
$$\psi(x,z).\psi(y,z).\vee\psi(z,x).\psi(z,y).\vee.z=x\vee z=(y).$$

5. *Note on Classes.* It may be suggested that modal propositions have some bearing on the customary treatment of classes through their determining functions. There are two related questions which arise in this connection; namely, whether there exist determining functions of the proper order, and whether there exist determining functions at all. We have suggested how, in certain cases, difficulties

as to order might be overcome. With regard to the second question, we wish to know that every class has a determining function; but this is not sufficient: we wish to know that every class must have a determining function. An illustration will make this clear. Suppose we have two classes α_1, α_2 and respective determining functions f_1, f_2, such that $(x).f_1(x) \supset f_2(x)$. Now we have occasion very often to make such assertions as,

(i) Every subclass of α_1 is a subclass of α_2;

and wish to express this by

$$\text{(ii)} \qquad (\phi):(x).\phi x \supset f_1(x). \supset .(x).\phi x \supset f_2(x).$$

Aside from difficulties of type, (ii) is not logically equivalent to (i) even if every subclass of α_1 has a determining function, unless it is also true that every subclass must have. But neglecting this point, let us suppose that there is some subclass β, of α_1, which has no determining function. Then, clearly, (ii) is not a right analysis of (i). But if we use a modal proposition in the analysis of (i), it seems to be unimportant whether every class has a determining function, or whether it must have; and we seem to be able to treat classes through functions whether classes have determining functions or not. For (i) may be expressed

$$\text{(iii)} \qquad (x).\bar{\phi}x \supset f_1(x). \supset .(x).\bar{\phi}x \supset f_2(x) \ ;$$

that is, being a function which determines a subclass of the class determined by f_1 entails being a function which determines a subclass of the class determined by f_2. Now, by hypothesis, β is a subclass of f_1 having no determining function; but (iii) is relevant to β, for it is nevertheless true that, if β had a determining function, that function would determine a subclass of f_2.

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