the future. It is evident that from the study of such systems associativity, important as it is, is of trifling importance in the general problems of structure. It may be then that the day will come when the matrix will be of only very limited importance in the study of structural physics, and the non-associative hypernumbers will give us the keys to the universe.

It is nevertheless a matter for congratulation to Professor Cullis that he is carrying out such a heavy piece of work as the present development of matrices, and it is to be hoped that he may bring it to a successful conclusion.

J. B. SHAW

STENSTRÖM ON THE 27 LINES

Synthetische Untersuchungen des Systems von 27 *Geraden einer Fläche dritter Ordnung.* By Olof Stenström. Upsala, Appelbergs Boktryckeri Aktiebolag, 1925. 128 pp.

It was a matter of course, in the history of geometry, that the cubic surface should be much studied. The zest with which many have pursued this study was not a matter of course but was largely due to the elegant configuration of the 27 lines on the surface. Since their fortunate discovery by Cayley and Salmon in 1849, they have been essential to nearly all investigations concerning it—investigations in which conspicuous names are Schläfli, Clebsch, Sylvester, Cremona, Sturm, Klein, Reye, Zeuthen, Schur, Henderson, and Baker. No decade has, indeed, been without notable work on the surface of third order. Since 1920 the contributions of two men deserve especial mention. The first was E. Stenfors,* who worked on the projective transformations of a Schläfli double-six into itself, as well as on like transformations of the whole system of lines, and their groups. Olof Stenström followed in 1925 with the booklet which is the subject of this review.

If, contrary to custom, a formal Table of Contents is included in a review, it may perhaps be justified by the fact that the book itself contains none, although such a table would aid both those who wish to read it and those who wish to learn more quickly of its content.

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* *Die Schläflische Konfiguration von zwölf Geraden einer Fläche dritter Ordnung,* 1921. *Über die Geradenkonfiguration einer Fläche dritter Ordnung, bzw. Klasse,* 1922. Both memoirs are contained in Series A, vol. 18, of that valuable periodical which the Union List of Serials names Suomalaisen Tiedeakatemian Toimituksia, but which the Revue Semestrielle designates—equally correctly—as Annales Academiae Scientiarum Fennicae. One who searches for a volume of this rather scarce journal wishes that there were agreement on nomenclature.

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	(The heading is omitted, but should follow line 2.)	
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From the title two things are clear: that Stenström follows the synthetic method which had its first great impetus in the study of the cubic surface with Cremona's memoir of 1868*; and that his researches concern the lines primarily, not the surface. Nevertheless, the Table of Contents indicates, among its wealth of subjects, three chapters (V, VI, and IX) in which the surface is treated also.

Any one of the 27 lines meets ten others in pairs of an involution. In the case of twelve lines, which happen to constitute a double-six, the involution is elliptic; on the other lines it is hyperbolic. Here we are considering, of course, the surface whose lines are real and distinct. This fact, early recognized by Klein and Zeuthen, is the basis for Stenström's detailed analysis of the various figures formed by the lines; as a result the geometry always alluringly intricate, becomes truly elaborate. Thus our author distinguishes three types of planes determined by lines of the system, four types of double-sixes, twelve of skew triples of lines, five of Steinerian triedral pairs, and so on. The dissection of the surface by means of each kind of double-six is displayed; in particular, it appears that all parabolic points lie within those ten triangles whose boundaries carry hyperbolic involutions.

The last chapter, devoted to the cross ratios of quadruples of planes through lines of the system, continues work begun by Kasner,† but brings

* *Mémoire de géométrie pure sur les surfaces du troisième ordre*, Journal für Mathematik, vol. 68.

† *The double-six configuration connected with the cubic surface and a related group of Cremona transformations*, American Journal of Mathematics, vol. 25 (1903).

results in a new and decidedly elegant form. Let us, for instance, consider the five planes of Type II—the type in which each of the three lines has a hyperbolic involution, with the intersecting lines outside the plane giving two pairs of points on each segment bounded by lines in the plane. Let the other four planes through each of the three lines have cross ratio λ_n; this will be the same for the three lines, but will differ from plane to plane. Then these five cross ratios satisfy the symmetric equation

$$(\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5)(1-\frac{1}{\lambda_1}-\frac{1}{\lambda_2}-\frac{1}{\lambda_3}-\frac{1}{\lambda_4}-\frac{1}{\lambda_5})+1$$
$$-\lambda_1-\lambda_2-\lambda_3-\lambda_4-\lambda_5=0.$$

A notable characteristic of Stenström's work is his ingenuity in notation, and in the use of diagrams. The capitalization of the usual a_n and b_n, in order to set off that double-six whose involutions are elliptic, was fairly obvious. Decidedly clever is the indication of degenerate double-sixes of Type IV, by means of the vertices and centers of two regular pentagons—in each pentagon six points for the six indices. This type of double-six

$$\begin{matrix} A_i & C_{np} & A_k & C_{mp} & A_l & C_{mn} \\ C_{kl} & B_m & C_{il} & B_n & C_{ik} & B_p \end{matrix}$$

obviously individuated by the indices i, k, l, "is represented by (one of) the ten triangles with an obtuse angle which can be formed from these points." The other three indices give an acute triangle in the other pentagon. 26 cases of degeneracy, caused by coincidence of conjugate lines in such double-sixes, are enumerated; and the triangles drawn in the pentagonal skeleton indicate at once the collapsed double-sixes, the coincident lines, the number and multiplicity of the lines and planes.

The pentagonal representation does duty only so long as the lines are real; admission of imaginary lines raises the number of types to 75. These 75 cubic surfaces are spread out on a family tree. From the common ancestor—the system of 27 real and distinct lines—one passes downward and to the right whenever lines come to coincidence, then upward, but still to the right, when they separate again and become imaginary. Thus the level on which the symbol for a particular system (or surface) is found shows at once the number of collapsed double-sixes.

All diagrams are decidedly neat, even if they lack the perfection of Henderson's plates* and the gaiety of Stenfors' black, red, and green pictures.

Proof-reading on Stenström's book was faulty. However, most of the errors were corrected in the copy supplied the reviewer, and it is to be hoped that these corrections will become part of the standard equipment of the book. One of them, indeed, is distressing (page 97, line 23), annulling a predicate and allowing the subject to survive as best it may.

In the excellent historical introduction we are puzzled by a reference to a "Probeartikel der Encyclopädie der Mathematischen Wissenschaften

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* *The Twenty-Seven Lines upon the Cubic Surface*, Cambridge Tracts, No. 13.

von 1896"; for the publication of the encyclopedia began in 1898, the article on cubic surfaces has not yet appeared, and Stenström's "Literaturverzeichnis" gives no item dated 1896. There is confusion (page 22) in the discussion of Schur quadrics associated with various types of double-sixes. "Eine *nicht nullteilige* Fläche zweiter Ordnung teilt den Raum in zwei Bereiche. . . . . Liegen (die Geraden des Paares) auf derselben Seite (der Fläche), so ist sie *nullteilig*."

The configuration which Stenström (page 26) names "Sturmsche Doppelfünfseite" is not that which Sturm himself describes.* A double-five satisfying Sturm's specifications, but not Stenström's, is

$$\begin{matrix} C_{23} & C_{45} & C_{26} & C_{34} & C_{56} \\ C_{46} & C_{36} & C_{35} & C_{25} & C_{24}. \end{matrix}$$

The reverse is true of

$$\begin{matrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56}. \end{matrix}$$

On page 37 it is stated that "einander nicht schneidende Gerade bleiben einander nicht schneidende oder fallen ganz und gar zusammen." Yet when, in

$$\begin{matrix} A_1 & A_2 & A_3 & C_{56} & C_{64} & C_{45} \\ C_{23} & C_{31} & C_{12} & B_4 & B_5 & B_6, \end{matrix}$$

corresponding pairs come to coincidence, all lines become concurrent, and A_1 and A_2 intersect without coinciding. This section could be stated more carefully. And what is the meaning of the assertion (page 63) that a double-six whose lines have become imaginary is "im ganzen betrachtet immer noch reell"?

Stenström devotes some attention to the Sylvester pentahedron, whose vertices are the double points of the cubic surface's Hessian; and he hints that in this matter he plans further investigation, which is to develop the relation between the system of lines and the pentahedron. It is to be hoped that we shall learn of the success of this attempt; for the present book establishes its author's ability very substantially to extend our knowledge in this field of geometry.

E. S. ALLEN

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* *Die* 27 *Geraden der cubischen Fläche,* Mathematische Annalen, vol. 23 (1884), p. 296.