CONCERNING CONNECTED AND REGULAR POINT SETS*

BY G. T. WHYBURN

In this paper an extension will be given of a theorem of the author's† which states that if A and B are any two points of a continuous curve M, and K denotes the set of all those points of M which separate‡ A and B in M, then $K+A+B$ is a closed set of points. It will be shown that if A and B are any two points of any connected and regular (connected im kleinen) point set M, and K denotes the set of all those points of M which separate A and B in M, then $K+A+B$ is a closed and bounded set of points. This extended theorem is applied to show that a simple continuous arc may be defined as a connected and regular point set which is irreducibly connected§ between some two of its points.

THEOREM 1. *If A and B are any two points of a connected and regular point set M, and K denotes the set of all those points of M which separate A and B in M, then $K+A+B$ is a closed and bounded set of points.*

PROOF. I shall first show that $K+A+B$ is closed. Suppose, on the contrary, that there exists a point P which does not belong to $K+A+B$ but which is a limit point of K. Then there exists a sequence of points $S=Y_1, Y_2, Y_3, \cdots$,

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† G. T. Whyburn, *Some properties of continuous curves*, this Bulletin, vol. 33 (1927), pp. 305–308.

‡ The points A and B of the connected point set M are said to be separated in M by the point X of M provided that $M-X$ is the sum of two mutually separated point sets containing A and B respectively.

§ A connected point set M is said to be irreducibly connected between two of its points A and B provided that no proper connected subset of M contains both A and B. See N. J. Lennes, American Journal of Mathematics, vol. 33 (1911), p. 308. See also Knaster and Kuratowski, *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2 (1921), p. 206.

belonging to K and having P as its sequential limit point. Each point Y_i of this sequence separates M into two mutually separated sets $M_a(Y_i)$ and $M_b(Y_i)$ containing A and B respectively. Either there exists a subsequence W of S such that if Y is any element of W, then $M_b(Y)$ contains at least one point of W, or there does not exist any such sequence.

CASE I. Suppose there exists a subsequence W of S such that if Y is any point of W, then $M_b(Y)$ contains at least one point of W. Let X_1 denote one of the elements of W. Then $M_b(X_1)$ contains at least one point of W. Let X_2 denote one such point of W which belongs to $M_b(X_1)$. Likewise $M_b(X_2)$ contains at least one point X_3 of W, and $M_b(X_3)$ contains at least one point X_4 of W, and so on. This process may be continued indefinitely, giving an infinite subsequence $V = X_1, X_2, X_3, \cdots$, of S. Now since for each positive integer i, $M_a(X_i)+X_i$ is connected and does not contain X_{i+1} [for X_{i+1} belongs to $M_b(X_i)$], therefore $M_a(X_i)+X_i$ is a subset of $M_a(X_{i+1})$. It follows that the points of the sequence V are all distinct, and hence P is the sequential limit point of this sequence.

Now let

$$E = \sum_{i=1}^{\infty} M_a(X_i), \qquad F = M - E.$$

Since for each positive integer i, $M = M_a(X_i)+X_i+M_b(X_i)$, and $M_a(X_i)+X_i$ is a subset of $M_a(X_{i+1})$, it is clear that $U=\sum_{i=1}^{\infty} X_i$ is a subset of E and that F is identical with the set of points common to all of the sets $M_b(X_i)$. Now no point of F, save possibly P (in case P belongs to M), is a limit point of E. For suppose some point Q of F, distinct from P, is a limit point of E. Then since Q is neither a point nor a limit point of U, there exists a circle C having Q as center and which neither contains nor encloses any point of U. But since M is regular at Q, and Q is a limit point of E, there exists a point Z of E which can be joined to Q by a connected subset I of M lying wholly within C. There exists an integer

k such that Z belongs to $M_a(X_k)$. But Q belongs to every set $M_b(X_i)$ and hence belongs to $M_b(X_k)$. But since I is connected and does not contain X_k, it follows that $M_a(X_k)$ and $M_b(X_k)$ are not mutually separated, contrary to hypothesis. Therefore no point of F, save possibly P, is a limit point of E. Furthermore, no point whatever of E is a limit point of F. For if Q is any point of E, then for some positive integer j, $M_a(X_j)$ contains Q, and since $M_b(X_j)$ contains F, and $M_a(X_j)$ and $M_b(X_j)$ are mutually separated, it follows that Q is not a limit point of F.

Now $M=E+F$, and if P does not belong to M, then E and F are mutually separated sets, contrary to the fact that M is connected. And if P does belong to M, then $M-P=E+(F-P)$ is the sum of two mutually separated sets E and $F-P$ containing A and B respectively. Hence P separates A and B in M and therefore belongs to K, contrary to supposition. Thus, in this case, the supposition that $K+A+B$ is not closed leads to a contradiction.

CASE II. Suppose S contains no subsequence W such that if Y is any element of W, then $M_b(Y)$ contains at least one point of W. Then no matter what subsequence W of S we take, W contains some point Y such that $M_a(Y)$ contains all the points of W except Y. Hence the sequence S itself contains a point X_1 such that $M_a(X_1)$ contains all the points of S except X_1. Let S_1 be the subsequence of S obtained by omitting the point X_1 from S. Then S_1 contains a point X_2 such that $M_a(X_2)$ contains all the points of S_1 except X_2. Let S_2 be the sequence obtained by omitting from S_1 the point X_2. Then S_2 contains a point X_3 such that $M_a(X_3)$ contains all the elements of S_2 except X_3. This process may be continued indefinitely, giving an infinite sequence of points $V=X_1, X_2, X_3, \cdots$, which is a subsequence of S. For each positive integer i, every point of the sequence S_i belongs to the set $M_a(X_i)$. Then since, for each i, $M_b(X_i)+X_i$ is connected and does not contain X_{i+1} [for X_{i+1} is an element of S_i, and hence belongs to $M_a(X_i)$], therefore $M_b(X_i)+X_i$ is a subset of $M_b(X_{i+1})$. It follows that the points of the sequence V are all distinct, and hence P is the sequential limit point of this sequence.

Let

$$E = \sum_{i=1}^{\infty} M_b(X_i), \qquad F = M - E.$$

Then by an argument similar to that given under Case I, with the sets $M_a(X_i)$ and $M_b(X_i)$ interchanged, it is shown that in this case the supposition that $K+A+B$ is not closed leads to a contradiction. Therefore the set $K+A+B$ is closed.

I shall now show that the set $K+A+B$ is bounded. If K exists at all, it is clear that there exists a point O not belonging to M. Let T be an inversion of the plane about the point O. As M is regular, then $T(M)$ also is regular. Now $T(K)$ is identically the set of points in $T(M)$ which separate $T(A)$ and $T(B)$ in $T(M)$. Therefore, by the proof given above, $T(K)+T(A)+T(B)$ is a closed set of points. Since this set of points does not contain O, there exists a circle $T(C)$ with O as center and neither containing nor enclosing any point of the set $T(K)+T(A)+T(B)$. Then since the exterior of the circle $T(C)$ is the image of the interior of the circle C, the set of points $K+A+B$ lies wholly within the circle C and therefore is bounded. This completes the proof of the theorem.

THEOREM 2. *If the connected and regular point set M is irreducibly connected between some two of its points A and B, then M is a simple continuous arc from A to B.*

PROOF. Let K denote the set of all those points of M which separate A and B in M. Then since M is irreducibly connected between A and B, it follows by a theorem due to Knaster and Kuratowski* that $K = M-(A+B)$. Hence $K+A+B=M$. But by Theorem 1, $K+A+B$ is closed and bounded. Therefore M is closed and bounded and irre-

* *Sur les ensembles connexes*, loc. cit., p. 219, Theorem 19.

ducibly connected between A and B, and hence satisfies all the conditions of Lennes'* definition of a simple continuous arc from A to B.

The chief interest in the above definition of a simple continuous arc lies in the fact that in its statement nothing is said concerning the closure or the boundedness of the set in question. This result for the case of the arc is related to results obtained by R. L. Wilder† for the case of a simple closed curve. Wilder found that definitions of a simple closed curve could be given in which the closed and bounded conditions were replaced, or modified, by adding the condition of regularity.

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NOTE ON THE FOURIER DEVELOPMENT OF CONTINUOUS FUNCTIONS‡

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If we have given a function, $f(t)$, which is continuous and periodic, this function is not necessarily developable in a Fourier series, but by a monotone change of variable it becomes so. We shall prove in fact the following theorem.

THEOREM. *If a function,* $f(t)$, *is continuous and periodic with period* $b-a$, *then there exists a monotone continuous function,* $t=t(\theta)$, *transforming the interval* $(0, 2\pi)$ *into the interval*

* N. J. Lennes, loc. cit. Hallett has shown that the condition of boundedness in Lennes' definition is superfluous. See G. H. Hallett, Jr., *Concerning the definition of a simple continuous arc*, this Bulletin, vol. 25 (1919), pp. 325–326. The same result was published two years later by Knaster and Kuratowski in their paper *Sur les ensembles connexes*, loc. cit., p. 224, Theorem 27.

† See an abstract of a paper *On the definition of simple closed curve*, this Bulletin, vol. 32 (1926), p. 123. See also p. 591.

‡ Presented to the Society, April 16, 1927.