ZEROS OF A FUNCTION AND OF ITS DERIVATIVE*

BY W. J. TRJITZINSKY

Macdonald has proved the following theorem.† Let
\[ f(z) = u(x, y) + iv(x, y) \]
be a function of \( z \) analytic throughout the interior of a single closed curve \( C \), defined by the equation
\[ |f(z)| = M, \]
where \( M \) is a constant. Let \( C \) be an ordinary curve in the sense that if \( \psi \) be the angle which the tangent to \( C \) makes with the \( x \)-axis, and if the point of contact of the tangent describes the curve \( C \), \( \psi \) will increase by \( 2\pi \). Then the number of zeros of \( f(z) \) in this region exceeds the number of zeros of the derivative \( f'(z) \) by unity. Our purpose is to generalize this result by showing the conclusion holds true even when \( M \) is not a constant, but a function \( M = M(x, y) \) analytic and greater than zero within sufficiently large intervals, and \( C \) is a curve single, closed, and ordinary (in the sense mentioned above), and satisfying the equation
\[ |f(z)| = M(x, y), \]
provided that the point \( (x = x(\theta), y = y(\theta)) \) describes \( C \) once when \( \theta \) changes from 0 to \( 2\pi \); where \( x(\theta) \) and \( y(\theta) \) are periodic functions satisfying the equations
\[ \begin{align*}
\frac{u(x, y)}{M(x, y)} &= \cos \theta, \\
\frac{v(x, y)}{M(x, y)} &= \sin \theta.
\end{align*} \tag{1} \]
The solvability of these equations implies that the Jacobian of the left members does not vanish identically.

On \( C \), let \( f(z) = M(x, y)e^{i\theta} \). Then there results a pair of equations (1) whose solutions are \( x = x(\theta) \) and \( y = y(\theta) \) of period \( 2\pi \) in \( \theta \). By hypothesis, among them there is at least one solution, say \( x = x(\theta), y = y(\theta) \), representing the curve \( C \), and the point \( (x = x(\theta), y = y(\theta)) \) describes \( C \) once as \( \theta \) varies from 0 to \( 2\pi \).

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Letting \( M(x(\theta), y(\theta)) = N(\theta) \), we observe that \( N(\theta) \) as well as \( N'(\theta) \) is of period \( 2\pi \), and for all \( \theta \), \( N(\theta) > 0 \), so that when \( z \) describes \( C \) and \( \theta \) varies from 0 to \( 2\pi \), the complex quantity \( N'(\theta) + iN(\theta) \) describes a closed curve entirely above the real axis; its modulus will return to the initial value and the variation of its argument will be zero. Now on \( C \) we have

\[
f(z) = N(\theta) \cdot e^{i\theta},
\]

\[
f'(z) = e^{i\theta} \cdot \frac{d\theta}{dz} \cdot \left[ N'(\theta) + iN(\theta) \right],
\]

\[
f''(z) = e^{i\theta} \cdot f\left\{ \left[ N'(\theta) + iN(\theta) \right] \cdot \left[ \frac{d^2\theta}{dz^2} + i\left( \frac{d\theta}{dz} \right)^2 \right] + \left[ N''(\theta) + iN''(\theta) \right] \left( \frac{d\theta}{dz} \right)^2 \right\}.
\]

The excess \( e \) of the number of zeros of \( f(z) \) over the number of zeros of \( f'(z) \) within \( C \) is

\[
e = \frac{1}{2\pi i} \int_C f''(z) \frac{dz}{f(z)} - \frac{1}{2\pi i} \int_C f'(z) \frac{dz}{f(z)}
\]

\[
= -\frac{1}{2\pi i} \int_C \frac{d^2\theta}{dz^2} \frac{dz}{(d\theta/dz)^2} - \frac{1}{2\pi i} \int_C d\theta \left[ \frac{N''(\theta)}{N(\theta)} + \frac{N'(\theta)}{N(\theta)} \right] - \frac{N'(\theta)}{N(\theta)} \left( \frac{d\theta}{dz} \right)^2 \frac{dz}{dz}.
\]

The quantity

\[
-\frac{1}{2\pi i} \int_C \frac{d^2\theta}{dz^2} \cdot dz = 1,
\]

as can be seen from Whittaker's* presentation of the theorem cited. Hence we have

\[\text{(*) Whittaker and Watson, Modern Analysis, 3d ed., p. 121.}\]
\[ e = 1 - \frac{1}{2\pi i} \int_C \left( \frac{dz}{d\theta} \right) \cdot \left[ \frac{N''(\theta) + iN'(\theta)}{N'(\theta) + iN(\theta)} - \frac{N'(\theta)}{N(\theta)} \right] d\theta, \]

since \( \theta \) varies from 0 to \( 2\pi \) when \( z \) describes \( C \). Moreover

\[ e = 1 - \frac{1}{2\pi i} \cdot \left\{ \log \left[ N'(\theta) + iN(\theta) \right] \right\}_0^{2\pi} + \frac{1}{2\pi i} \left[ \log N(\theta) \right]_0^{2\pi}. \]

We know that \( N(\theta) \) is real; hence \( \left[ \log N(\theta) \right]_0^{2\pi} = 0 \). On the other hand, the variation of the argument of \( \left[ N'(\theta) + iN(\theta) \right] \) as \( \theta \) changes from 0 to \( 2\pi \) is zero, so that \( \log \left[ (N'(\theta) + iN(\theta)) \right]_0^{2\pi} = 0 \). Hence \( e = 1 \). This proves the theorem.

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LINEAR INEQUALITIES IN GENERAL ANALYSIS*

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1. Introduction. In his studies in general analysis, E. H. Moore has developed† a theory of the linear functional equation

\[ \xi + J\kappa \xi = \eta. \]

Here \( \xi \) and \( \eta \) denote functions (the latter given, the former to be determined) belonging to a class \( M \) of real-valued functions on a general range \( \Psi \). The kernel function \( \kappa \) belongs to a class \( \kappa \) which is well defined in terms of the fundamental class \( M \). A sufficient foundation for the theory is laid by means of postulates upon the class \( M \) and the functional operation \( J \).

The purpose of the present paper is to consider the linear inequality

\[ (1) \quad \xi + J\kappa \xi > 0, \]

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