

GENERALIZATIONS OF THE THEOREM OF
FERMAT AND CAUCHY ON
POLYGONAL NUMBERS*

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1. *Introduction and Summary.* We seek the least l such that every integer $A \geq 0$ is a sum of l values of

$$(1) \quad p_{m+2}(x - k) = \frac{1}{2}(x - k)[m(x - k - 1) + 2]$$

for integers $x \geq 0$. When $k=0$, (1) is a polygonal number of order $m+2$, and Fermat stated that $l=m+2$. This was first proved by Cauchy, who found that all but four of the $m+2$ polygonal numbers may be taken as 0 or 1. A simpler proof was given by the writer in this Bulletin (vol 33 (1927), pp. 719-726) that paper will be cited as I.

When all but four of the l values of (1) are 0 or 1, we shall prove here that

$l = m - 1$ if $k = 1$, $m \geq 7$; $l = m - 2$ if $k = 2$, $m \geq 8$, or $k \geq 3$, $m \geq 7$; $l = 6$ if $k = 1$, $m = 6$, or $k = 2$, $m = 7$; $l = 5$ if $k = 1$, $m = 4$ or 5, or if $k \geq 2$, $m = 6$; $l = 4$ if $k \geq 1$, $m = 3$, or $k \geq 2$, $m = 4$ or 5.

When all but s of the l values of (1) are 0 or 1, where $5 \leq s \leq l$, we shall prove that, for every $k \geq 1$,

$$l = m - 2 \quad \text{if } m > 7, \quad l = s \quad \text{if } m \leq 7.$$

But A is a sum of l values of (1) if and only if †

$$(2) \quad 8mA + l(m - 2)^2 = \sum_l (2mx + 2 - m - 2mk)^2,$$

summed for l integers $x \geq 0$. We saw that $l=4$ when $k=1$, $m=3$, and when $k=2$, $m=4$ or 5. Hence (2) shows that for every integer $A \geq 0$,

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† This implies the like formula with k replaced by any larger integer $k+p$. The values $x \geq p$ give the same squares as occur in (2) for $x \geq 0$. Hence we use the least permissible k .

$$(3) \quad \begin{aligned} 24A + 4 &= \sum_4 (6x - 7)^2, & 40A + 36 &= \sum_4 (10x - 23)^2, \\ 8A + 4 &= \sum_4 (4x - 9)^2. \end{aligned}$$

By Theorem 6 and (2), we have

$$(4) \quad 3A + 4 = \sum_4 (3x - 16)^2 \quad \text{if } A \not\equiv 4 \pmod{8}.$$

For $s=5$, $m \leq 7$, we saw that $l=5$. Then (2) for $k=1$ is

$$(5) \quad 8mA + 5(m-2)^2 = \sum_5 (2mx + 2 - 3m)^2.$$

For $m=3$ this result is obtained by adding 1 to each member of (3₁) and is discarded. When m is 4 or 6, we cancel factors 4 or 16. We get

$$(6) \quad 8A + 5 = \sum_5 (4x - 5)^2, \quad 40A + 45 = \sum_5 (10x - 13)^2,$$

$$(7) \quad 3A + 5 = \sum_5 (3x - 4)^2, \quad 56A + 125 = \sum_5 (14x - 19)^2.$$

The results obtained when $s=4$, $l=5$ are consequences of these.

When $k=1$, $m \geq 8$, Theorem 2 states that $l=m-2$ except for $A=54m+16$. This is a remarkable fact since the single exceptional value may be made as large as we please by taking m sufficiently large. Again, if $k=5$, $m=6$, we find that $l=4$ fails first for $A=980$; by increasing k , the first failure will occur for a value of A exceeding any assigned number.

In later papers I shall prove that the value of l obtained when $s=3$ is never less than the minimum l found here, and I shall give an improved method valid when $s>4$, but not when $s=4$.

2. *Formulas.* Give (1) the notation $\frac{1}{2}mx^2 + \frac{1}{2}nx + c$. Thus

$$(8) \quad n = 2 - m - 2mk, \quad c = \frac{1}{2}m(k^2 + k) - k.$$

For $k \geq 1$ the reduced minor conditions in I now follow from

$$(9) \quad A \geq 4c + 4E, \quad A \geq 4c + \frac{2}{3}m.$$

For, from the inequality (9₂) we deduce (11) of I, since the sum $(-n)(2m+n)/m$ of its last two terms is negative, because $-n$ is positive and $2m+n$ is negative. By (8₁), we may write (4) of I in the form

$$(10) \quad A = mg + 4c + b + r, \quad g = \frac{1}{2}(a - b - 2kb).$$

3. *Excess E when k=1.* The values of (1) for $x \geq 0$ are $m-1$ and all the polygonal numbers of order $m+2$.

TABLE II.

SUMS BY FOUR OF $m-1$ AND POLYGONAL NUMBERS

0-4, $m-1$, m , $m+1-5$, $2m-2$, $2m-1$, $2m$, $2m+1-6$, $3m-3$, $3m-2$, $3m$, $3m+1$, $3-7$, $4m-4$, $4m-1$, $4m+2-8$, $5m+1$, 2, 4, 5, 7, 8, $6m$, $6m+3-9$, $7m+3-9$, $8m+2-10$, $9m+1$, 4, 7-10, $10m+5-11$, $11m+4-9$, 11, $12m+3$, 4, 6-12, $13m+2$, 5, 7-12, $14m+6-12$, $15m+6-9$, 11-13, $16m+5-13$, $17m+4$, 5, 7-13, $18m+3$, 6, 7, 9-14, $19m+8$, 9, 11-14, $20m+7$, 10-14, $21m+7-13$, 15, $22m+6-15$, $23m+5$, 6, 8, 9, 11-15, $24m+4$, 7, 10-16, $25m+9-13$, 15, 16, $26m+8$, 10, 11, 13-16, $27m+9$, 11-16, $28m+8-17$, $29m+7-13$, 15-17, $30m+6$, 7, 9, 10, 12-17, $31m+5$, 8, 11-17, $32m+10-18$, $33m+9$, 10, 12, 13, 15-18, $34m+12-18$, $35m+11-17$, $36m+9-19$, $37m+8-13$, 15-19, $38m+7$, 8, 10, 11, 13-15, 17-19, $39m+6$, 9, 12-17, 19, $40m+11$, 12, 14-20, $41m+10$, 13, 15-20, $42m+13-20$, $43m+12-17$, 19, 20, $44m+11-20$, $45m+10-13$, 15-21, $46m+9-21$, $47m+8$, 9, 11-17, 19-21, $48m+7$, 10, 12-21, $49m+12$, 13, 15-21, $50m+11$, 14-22, $51m+13-17$, 19-22, $52m+13-22$, $53m+12$, 13, 15-21, $54m+17-19$, 21, 22.

If $m \geq 6$, $E(9m+6) = 2$, since neither $9m+5$ nor $9m+6$ is equal to a number of Table II. If $m=5$, Table II lacks $17 = 2m+7 = 3m+2 = 4m-3$. If $m=4$, it lacks $26 = 4m+10 = 5m+6 = 6m+2 = 7m-2$. Also,* $E(54m+16) = m-5$. Hence E is not smaller than the value shown in

THEOREM 1. For $k=1$, $E = m-5$ if $m \geq 7$, $E = 2$ if $m = 6$, $E = 1$ if $m = 4$ or 5, $E = 0$ if $m = 3$.

By (10) we require that the values of $b+r$ shall include a complete set of residues modulo m when r takes the values 0, 1, \dots , E , and b takes certain consecutive odd values. When $m=3$ or $m \geq 5$, this will be true if b takes the values

* And independently of Table II, since the only partitions of 53 into 0, 1, 3, 6, 10, 15, 21, 28, 36, 45 are $1+1+6+45$, $1+1+15+36$, $1+6+10+36$, $1+3+21+28$, $0+10+15+28$, $1+10+21+21$.

$\beta, \beta+2, \beta+4$, whence $d=6$. But when $m=4$, we need only $\beta, \beta+2$, whence $d=4$. Conditions (9) hold if $A \geq 8m$.

Let $m \geq 7$. By (6)–(15) of I,

$$\begin{aligned} U &= 24mA - 63m^2 + 12m + 36, & V &= 2mA - m^2 + 6m + 4, \\ P &= 7m + 2, & W &= 3mA - 14m^2 - 2m + 4, \\ F &= m^2A^2 - 74m^3A - 28m^2A + 193m^4 + 70m^3 - 68m^2 \\ & & & - 24m > 0. \end{aligned}$$

The last evidently holds if

$$(11) \quad A \geq 74m + 28.$$

Next, let $m=6$. Then

$$\begin{aligned} U &= 36(4A - 60), & V &= 12A - 8, & P &= 44, & W &= 18A - 512, \\ F &= 36[(7A - 88)^2 - (12A - 8)(4A - 60)] \\ & & & & & & & = 36(A^2 - 480A + 7264), \end{aligned}$$

and $F > 0$ if $A \geq 465$ and hence if (11) holds.

If $m \geq 6$, Theorem 1 now follows from Lemma 3 of I and the following lemma.

LEMMA 1. *If $m \geq 5$ in Table II, $E(A) \leq m-5$ when A is between any consecutive blocks, while $E(A) \leq 2$ when A is in any block.*

For, the difference between consecutive entries in any block is always ≤ 3 . If r is the term free of m in the leader $qm+r$ of any block, then $r+4$ is found to be the term free of m of a number of Table II within the preceding block. Hence $qm+r-1$ is the sum of $m-5$ and the number $(q-1)m+r+4$ in the table.

When $m=5$, the last two sentences hold after we suppress from Table II all entries down to and including the last entry which differs by 3 from the next entry of the block. The only exception is the leader $9m+7$, while $9m+6$ exceeds $8m+10$ by 1. Hence $E(A) \leq 1$ if $A \leq 54m+22$.

$$\text{For } m=5, d=6, n=-13, c=4,$$

$$\begin{aligned} U &= 120A - 1479, & V &= 10A - 1, & P &= 37, & W &= 15A - 356, \\ F &= 25A^2 - 10150A + 126685 > 0 \text{ if } A > 393 = 74m + 23. \end{aligned}$$

These facts with Lemma 3 of I prove Theorem 1 for $m = 5$.

$$\text{For } m = 4, d = 4, n = -10, c = 3,$$

$$U = 4(24A - 231), \quad V = 4(2A - 1), \quad P = 14, \quad W = 12A - 140,$$

$$F = 16[(7A - 37)^2 + (2A - 1)(24A - 231)] \\ = 16(A - 16)^2 + 16 \cdot 882 > 0.$$

Finally, let $m = 3$. Then $d = 6, n = -7, c = 2$,

$$U = 9(8A - 55), \quad V = 6A + 1, \quad P = 23, \quad W = 9A - 128,$$

$$F = 9[(7A - 42)^2 - (6A + 1)(8A - 55)] \\ = 9(A^2 - 266A + 1819) > 0,$$

if $A \geq 259$. For $A < 259$, Theorem 1 was verified by Tables I and II.

4. THEOREM 2. *If $k = 1, m \geq 8, E(A) \leq m - 6$ except for $A = 54m + 16$.*

Within every block of Table II the difference of any two consecutive entries is ≤ 3 . If f is the term free of m in the leader $qm + f$ of any block having $q \neq 16, 46, 52, 54$, then $f + 5$ is that of a number t occurring explicitly in the table. Hence $qm + f - 1$ is the sum of $m - 6$ and $t = (q - 1)m + f + 5$. For $q = 16, 46$, and 52 , a like result holds if we replace $f + 5$ by $f + 6$ and hence replace $m - 6$ by $m - 7$. Hence the theorem is true for $A < 54m + 16$. For $54m + 17 \leq A \leq 199m + 37$, it is true by Lemmas 3 and 4 of I.

For $A \geq 199m + 38, E = m - 6$, we have $d = 8$,

$$U = 24mA - 63m^2 + 12m + 36, \quad V = 2mA - m^2 + 8m + 4,$$

$$P = 11m + 2, \quad W = 3mA - 23m^2 - 4m + 4,$$

$$F = m^2A^2 - 200m^3A - 48m^2A + 562m^4 - 84m^3 \\ - 264m^2 - 48m.$$

Then $F > 0$, in fact for $A = 198m + 44 + t, t \geq 0$.

5. *Excess E_s when $k=1$.*

THEOREM 3. *When $k=1$, $E_s=0$ if $m \leq 7$, $s \geq 5$, and if $m > 7$, $s \geq m-2$, while $E_s = m-s-2$ if $m > 7$, $5 \leq s \leq m-2$.*

If $m \leq 5$, this follows from Theorem 1. If $m > 7$, $E_5 \leq m-7$ by Theorem 2 and the fact that $54m+16$ is the sum of the polygonal number $3m+3$ and the entry $51m+13$ of Table II. Since the polygonal numbers >1 exceed $m-1$, the summands yielding $m-2$ are all 1, whence $E_s(m-2) = m-2-s$. Hence $E_5 = m-7$. If $s > 5$, regard $s-5$ of the values 1 as polygonal numbers; hence $E_s \leq m-7-(s-5)$. This proves Theorem 3 except for $m=6$ and 7.

For $m=6$ or 7, we seek a constant C_m such that $E_4(A) \leq 1$ for every $A \geq C_m$. When r takes the values 0 and $E=1$, and b takes 3 or 4 consecutive odd values according as $m=6$ or $m=7$, the values of $b+r$ include a complete set of residues modulo m , whence $d=6$ or 8, respectively.

In the discussion in § 3 for $m \geq 7$, we had $d=6$, $E=m-5$. Hence it is valid here for $m=6$, $d=6$, $E=1$, and shows that $E_4(A) \leq 1$ if $A \geq 74m+28$. Likewise, the work at the end of § 4 is valid here for $m=7$, $d=8$, $E=1$, and shows that $E_4(A) \leq 1$ if $A \geq 198m+44$.

It remains to treat the values of A below these two limits. From Table II we readily deduce a list of the sums by five of $m-1$ and polygonal numbers and then verify for $m=6$ and $m=7$ that the list includes all positive integers $\leq 54m+25$. From thence to $74m+28$, $E_4(A) \leq 1$ if $m=6$ or 7 by Lemma 3 of I. If $m=7$ and $74m+20 \leq A \leq 199m+37$, $E_4(A) \leq 1$ by Lemma 4 of I. This completes the proof of Theorem 3.

6. *Excess E when $k=2$.* The values of (1) for $k=2$, $x \geq 0$ are $m-1$, $3m-2$, and all the polygonal numbers of order $m+2$. Evidently $E(m-2) = m-6$. For $m=7$ the sums by four of the values mentioned were tabulated to 395; the only consecutive integers missing from the list are 393-4; hence $E(394) = 2$, $E(A) \leq 1$ if $A < 394$. For $m=6$, $E(60) = 1$. Hence E is not smaller than the value shown in

THEOREM 4. For $k=2$, $E=m-6$ if $m \geq 8$, $E=2$ if $m=7$, $E=1$ if $m=6$, $E=0$ if $m=3, 4$, or 5 . If $m=7$, $E(A) \leq 1$ except for $A=394$.

Conditions (9) are satisfied if $A \geq 16m$. For $r=0, 1, \dots$, E , we require that $b+r$ shall include a complete set of residues modulo m . When $m \geq 8$, this will be true if b takes the values $\beta, \beta+2, \beta+4, \beta+6$; when $m=5$, also $\beta+8$; when $m=6$ or 7 , only the first three.

First, let $m \geq 8$. Then $d=8$,

$$\begin{aligned} U &= 24mA - 135m^2 + 36m + 36, & P &= 9m + 2, \\ V &= 2mA - m^2 + 8m + 4, & W &= 3mA - 27m^2 + 4, \\ F &= m^2A^2 - 112m^3A - 40m^2A + 706m^4 + 188m^3 - 152m^2 \\ & & & - 48m. \end{aligned}$$

Evidently $F > 0$ if $A \geq 112m+40$. For smaller A 's exceeding $54m+16$, Lemmas 3 and 4 of I show that $E(A) \leq m-6$ if $m \geq 7$.

LEMMA 2. If $m \geq 7$ in Table II, $E(A) \leq m-6$ except for $A=9m+6, 20m+9$, and $54m+16$.

From each block we suppress all entries down to and including the last entry which differs by 3 from the next entry. Proceed as in § 4. We now have the further exceptions $q=9$ and 20 . Also

$$\begin{aligned} 9m + 3 &= m - 6 + t_1, & 9m + 5 &= 1 + t_2, \\ 20m + 8 &= m - 6 + t_3, & 54m + 15 &= m - 6 + t_4, \end{aligned}$$

where the t_i occur in the table.

For $m \geq 8$, Theorem 4 now follows since

$$\begin{aligned} 9m + 6 &= 1 + 2(3m + 3) + 3m - 2 + 1, & 20m + 9 &= m + 2 \\ & & & + 6m + 4 + 10m + 5 + 3m - 2, \end{aligned}$$

$$54m + 16 = 2(3m + 3) + 45m + 10 + 3m - 2 + 2.$$

For $m=7$, $E=2$, $d=6$, $P=37$,

$$\begin{aligned} U &= 168A - 6327, & V &= 14A - 3, & W &= 21A - 962, \\ F/7 &= 7A^2 - 826A + 131149, \end{aligned}$$

and $F > 0$ for every $A > 0$. But if we attempt to use $E = 1$, we have $d = 8$; the discussion for $m \geq 8$ now applies also for $m = 7$ except when $A = 54m + 16 = 394$. This proves both parts of Theorem 4 if $m = 7$.

Next, let $m = 6$. Then $E = 1$, $d = 6$, $P = 32$,

$$\begin{aligned} U &= 9(16A - 512), \quad V = 4(3A + 1), \quad W = 18A - 704, \\ F/36 &= (7A - 116)^2 - (3A + 1)(16A - 512) = (A - 52)^2 \\ &\quad + 11264. \end{aligned}$$

The minor conditions in I are all satisfied if $A \geq 44$. For $m = 6$, Table II includes all numbers less than 44 except $33 = 1 + 2(3m - 2)$.

Let $m = 5$. Then $E = 0$, $d = 10$, $P = 67$,

$$\begin{aligned} U &= 120A - 3159, \quad V = 10A + 9, \quad W = 15A - 956, \\ F/25 &= (A - 703)^2 - 457878 > 0 \text{ if } A \geq 1380. \end{aligned}$$

It was verified that every integer ≤ 1380 is a sum by four of 4, 13, and polygonal numbers of order 7, whence $E = 0$.

Let $m = 4$. Then $n = -18$, $c = 10$, $A = 2a - 9b + 40$. Our general method requires that a and b be odd and applies only when A is odd. By (10) with $r = 0$, $b \equiv A \pmod{4}$. Hence $d = 4$, $P = 6$,

$$\begin{aligned} U &= 4(24A - 495), \quad V = 4(2A + 1), \quad W = 4(3A - 63), \\ F/16 &= (7A - 61)^2 - (2A + 1)(24A - 495) = A^2 + 112A \\ &\quad + 4216, \end{aligned}$$

whence $F > 0$ for every $A \geq 0$. The minor conditions in I are all satisfied if $A \geq 28$. For $m = 4$, Table II includes all integers ≤ 28 except $26 = m + 2 + 2(3m - 2)$.

Next, let A be even. Since b must be even, our previous method is not applicable.

LEMMA 3.* *If $b = 2B$, a is even, $a - B^2$ is a sum of three squares, and*

* Legendre, *Théorie des Nombres*, ed. 3, II, No. 628, with omission of denominator 2 in No. 629.

$$(12) \quad 4B^2 + 2B + 1 > 3a,$$

then $a = \sum \alpha^2$, $b = \sum \alpha$ have solutions $\alpha, \beta, \gamma, \delta$ in integers ≥ 0 .

Write $A = 2S$. Then $a = S + 9B - 20$. Inserting this value of a into (12) and multiplying by 16, we see that (12) holds if

$$(13) \quad 8B > 25 + u^{1/2}, \quad u = 48S - 351 \geq 0.$$

Similarly, $B^2 < a$ if

$$(14) \quad 2B < 9 + v^{1/2}, \quad v = 4S + 1.$$

Take $B \equiv 2 - S \pmod{4}$. Then $a \equiv 2 \pmod{4}$, $a - B^2 \equiv 1$ or $2 \pmod{4}$, according as B is odd or even. Hence $a - B^2$ is a sum of three squares, and all conditions in Lemma 3 are satisfied.

To insure the choice of B as a prescribed residue modulo 4, we require that the difference between the limits for B , determined by (13) and (14), shall exceed 4. The resulting inequality reduces as usual to

$$S^2 - 340S + 2440 > 0, \quad S \geq 333.$$

It remains only to verify Theorem 4 for $m=4$ when A is even and < 666 . This was done above when $A \leq 28$. Table II includes all integers from 28 to 238 except 236; Table I includes all from 235 to 666.

7. *Excess when $k \geq 3$.* Evidently $E(m-2) = m-6$. If $m=7$,

$$394 = 54m + 16 = 6m - 3 + 2(10m + 5) + 28m + 8 + 1.$$

Hence Theorem 4 implies the cases $m \geq 7$ of

THEOREM 5. For $k \geq 3$, $E = m - 6$ if $m \geq 7$, $E = 1$ if $m = 6$.

That $E = 1$ if $m = 6$ follows from Theorem 4 and a result to be proved in a later paper on extended polygonal numbers. It was verified that 116 (or 980) is the least positive integer which is not a sum by four of the values of (1) for $x \geq 0$, $k=4$ (or $k=5$). For $x \geq 0$, the values > 1 of (1) are all $\leq m-1$. Hence the proof in § 5 shows that Theorem 3 holds also when $k > 1$.

8. THEOREM 6. *If $m=6$, $k=5$, $E(A)=0$ except for $A \equiv 4 \pmod{8}$.*

If A is odd, our general method applies with

$$d = 6, U = 144(A - 119), V = 12A + 16, P = -4.$$

Then (14) and hence (13) of I holds. Also, (9) hold if $A \leq 344$, which is below the limit 980 for which $E=0$ (§ 7).

Next, let $A \equiv 2 \pmod{4}$. Take $b=2B$ and determine B modulo 3 to make a an integer. Then $a \equiv 2 \pmod{4}$. Apply Lemma 3. Then (12) holds and $B^2 < a$ if

$$4B > 31 + u^{1/2}, \quad 3B < 32 + v^{1/2}, \quad u = 4A - 403, \\ v = 3A + 4.$$

The difference between these limits for B exceeds d if

$$(15) \quad 4v^{1/2} - 3u^{1/2} > R = 12d - 35.$$

Here $d=3$ and (15) holds for every A such that $u > 0$.

Finally, let $A \equiv 0 \pmod{8}$. Take an odd B such that a is an integer, whence $d=6$. Then $a \equiv 4 \pmod{8}$ and Lemma 3 applies. The square of (15) holds if

$$(A - 491)^2 > 65712, \quad A \geq 748.$$

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