A RATIONAL NORMAL FORM FOR CERTAIN QUARTICS*

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A quartic equation with rational coefficients, irreducible in the field R of rational numbers, will have for its Galois group (for the field R) one of the following:

I. The symmetric group of order 24.

II. The alternating group of order 12.

III. A group of order 8 such as the one whose substitutions are 1, (12), (34), (13)(24), (12)(34), (14)(23), (1324), (1423).

IV. A cyclic group of order 4 such as 1, (1324), (12)(34), (1423).

V. The 4-group 1, (12)(34), (13)(24), (14)(23).

It is the purpose of this note to give what seems to be a more direct proof of a theorem of Wiman:†

THEOREM. *Any rational quartic, irreducible in R, whose Galois group for R is either* III, IV, *or* V *above, can be transformed by a rational Tschirnhausen transformation into the form $y^4+py^2+q=0$, where p and q are rational.*

It will then follow as a corollary that p and q can be made integral; this can be accomplished by a simple additional transformation.

We may consider the given quartic in the reduced form

$$x^4 + a_2x^2 + a_3x + a_4 = 0, \quad (a_1 = s_1 = 0, a_3 \neq 0). \tag{1}$$

Apply the quadratic transformation

$$y = x^2 + k_1x + k_2. \tag{2}$$

Now the transformed equation will lack its second and fourth terms provided $\sum y=0$, $\sum y^3=0$.‡ The first of these equa-

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‡ The summations extend over the 4 roots of (1).

tions, by (2), will be satisfied if $k_2 = a_2/2$. To obtain $\sum y^3$ we cube both sides of (2), sum over the four roots of (1), and use Newton's relations applied to (1), namely

$$(3)\qquad \begin{cases} s_2 = -2a_2, \quad s_3 = -3a_3, \quad s_4 = 2a_2^2 - 4a_4, \\ s_5 = 5a_2a_3, \quad s_6 = 6a_2a_4 - 2a_2^3 + 3a_3^2. \end{cases}$$

The equation $\sum y^3 = 0$ then takes the form

$$(4)\qquad a_3k_1^3 - (a_2^2 - 4a_4)k_1^2 - 2a_2a_3k_1 - a_3^2 = 0.$$

Now it is known* that if x_1, x_2, x_3, x_4 denote the roots of (1), then $y_1 = x_1x_2 + x_3x_4$, $y_2 = x_1x_3 + x_2x_4$, $y_3 = x_1x_4 + x_2x_3$ are the roots of the resolvent cubic

$$(5)\qquad y^3 - a_2y^2 - 4a_4y + 4a_2a_4 - a_3^2 = 0.$$

And if the group of (1) is III, IV, or V, equation (5) will have at least one rational root. In this case (4) will also have a rational root, since the roots of (4) can be expressed rationally in terms of those of (5). It is easily verified that they are

$$(6)\qquad \frac{-a_3}{a_2 - y_1}, \quad \frac{-a_3}{a_2 - y_2}, \quad \frac{-a_3}{a_2 - y_3}.$$

These expressions for the roots of (4) were not, however, by any means obvious; the problem was to exhibit them in some form so that it would be possible to show that (4) has a rational root provided the group of (1) is III, IV, or V. For using this rational value of k_1, together with $k_2 = a_2/2$, we have exhibited the rational transformation which leads to the result of the theorem.

Bucht† has pointed out, and it is not difficult to show that if the group of (1) is V, the q of the transformed equation will be not only rational, but the square of a rational number.

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* Dickson, *Elementary Theory of Equations*, 1914, p. 39.

† Arkiv för Matematik, Astronomi, och Fysik, vol. 6 (1910–11), No. 30.