Hence $\lim_n c_n$ and $\lim_n b_n$ exist and are equal, i.e. $\lim_n \overline{\underline{\lim}}_m a_{mn}$ exists. From the symmetry of the condition, we conclude that $\lim_m \overline{\underline{\lim}}_n a_{mn}$ exists also. The identity of the two limits is then a consequence of the condition of our theorem and Statement A.

We note finally that the Cauchy condition for convergence of the double limit, $\lim_{mn} a_{mn}$, is the special case of our condition in which $m_{\epsilon n_1}$ and $n_{\epsilon m_1}$ are independent of n_1 and m_1 respectively, and can therefore be taken as m_ϵ and n_ϵ, respectively.

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ON BOUNDED REGULAR FRONTIERS IN THE PLANE*

BY W. A. WILSON

1. *Introduction.* The term *regular frontier* has been introduced by P. Urysohn† to designate a continuum which is the frontier of two or more components of its complement. Regular frontiers in the plane have been discussed by various authors. A. Rosenthal‡ has shown that a continuum which is the union of two bounded continua that are irreducible between the same pair of points and have no other common points is a regular frontier. R. L. Moore§ has given necessary and sufficient conditions that a bounded continuum be a regular frontier whose complement has exactly two components. C. Kuratowski|| has given necessary conditions for a continuum to be a regular frontier which is the frontier of *every* component of its complement.

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† P. Urysohn, *Mémoire sur les multiplicités Cantoriennes*, Fundamenta Mathematicae, vol. 7, p. 98.

‡ A. Rosenthal, *Teilung der Ebene durch Irreduzible Kontinua*, Sitzungsberichte der Münchener Akademie, 1919.

§ R. L. Moore, *Concerning the common boundary of two domains*, Fundamenta Mathematicae, vol. 6, pp. 203–213.

|| C. Kuratowski, *Sur les coupures du plan*, Fundamenta Mathematicae, vol. 6, pp. 130–145.

In the present article Rosenthal's theorem is generalized to cover the case of two continua having a finite or enumerably infinite set of closed sets in common and irreducible between each pair. With this result and a theorem by the author† given elsewhere, it is possible to formulate necessary and sufficient conditions for a bounded continuum to be a regular frontier. The principal theorems are to be found in §§5–7.

2. *Notation.* Besides the ordinary notation of the aggregate theory the following special notation and terminology will be used.

The whole plane will be denoted by Z. If A is a sub-set of C, the set of inner points of A relative to C will be denoted by A.* If A and B are two sets in the plane Z without common points and C separates A from B (i.e., $C \cdot A = C \cdot B = 0$ and every continuum in the plane which contains points of both A and B also contains one or more points of C), we say that C is an $S(A, B)$. If F is a frontier set, a component of $Z - F$ which has the frontier F will be called a *principal* component.

The statement "C is a continuum irreducible between the sets A and B" will apply not only to the case that $A + B \subset C$, but also to the case that C is irreducible between each point of $C \cdot A$ and each point of $C \cdot B$.

3. *Some Auxiliary Theorems.* Let $F = H + K$ be the union of two bounded continua and let $H \cdot K$ be the sum of a finite set of closed sets $\{\alpha\}$, such that both H and K are irreducible between each pair. The following properties of F are either well known or so easily established that the proofs are omitted.

(a) *If the number of sets* $\{\alpha\}$ *is greater than two, both* H *and* K *are indecomposable.*

(b) *If* C *is a sub-continuum of* H *or of* K, $F - C$ *is connected.*

(c) $H^* = F - K$ *and* $K^* = F - H$ *are connected; also* $\overline{H}^* = H$ *and* $\overline{K}^* = K$.

† W. A. Wilson, *On irreducible cuts of the plane between two points.* (To appear soon in Annals of Mathematics.)

If F lies in a plane Z, certain properties expressing relations between F and the rest of the plane can be proved without difficulty.

(d) *If $H \cdot K$ has at least n components, there are at least n components of $Z-F$ which have frontier points on both H^* and K^*. If $H \cdot K$ has an infinity of components, the components of $Z-F$ having frontier points on both H^* and K^* are enumerably infinite.*

These statements are proved by adding to H and K those components of $Z-F$ which do not have frontier points on both H^* and K^* and applying a theorem of S. Straszewicz.†

(e) *There is a bounded continuum P which is an irreducible $S(H^*, K^*)$ and contains $H \cdot K$.*

To establish this we observe that, since H^* and K^* are connected, there is a bounded continum Q which is an $S(H^*, K^*)$ by a theorem of Knaster and Kuratowski.‡ But by a theorem of Mazurkiewicz,§ Q contains a subcontinuum P which is an irreducible $S(H^*, K^*)$. Moreover $P \supset H \cdot K$, since every point of $H \cdot K$ is a common limiting point of H^* and K^*.

It may be added that every bounded irreducible $S(H^*, K^*)$ is a continuum containing $H \cdot K$.

4. Lemma. *Let $F=H+K$ be the union of two bounded continua and let $H \cdot K$ be the sum of a finite set of closed sets $\{\alpha\}$ between each pair of which both H and K are irreducible. Let R be a component of $Z-F$ such that $\overline{R}$ contains a bounded continuum containing points of two or more sets α but no other points of F. Then F is the frontier of R.*

Proof. The principles of inversion permit us to demonstrate the lemma on the assumption that R is bounded. Let

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† S. Straszewicz, *Über die Zerschneidung der Ebene*, Fundamenta Mathematicae, vol. 7, p. 174.

‡ B. Knaster and C. Kuratowski, *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2, p. 236.

§ S. Mazurkiewicz, *Sur un ensemble G*, etc., Fundamenta Mathematicae, vol. 1, p. 63.

C be the continuum referred to above. Then R contains a closed polygon J (whose interior is I) so large that $C-C\cdot I$ has no component containing points of more than one α. This is easily seen since the number of sets α is finite. On the other hand, if α_1 and α_2 are two sets met by C, there are sub-continua A_1 and A_2 of $C-C\cdot I$ irreducible between α_1 and α_2, respectively, and J, and $A_1\cdot A_2=0$. Let M be an arc of J irreducible beween A_1 and A_2. Then $Q=A_1+M+A_2$ is irreducible between α_1 and α_2 and does not meet $F-(\alpha_1+\alpha_2)$.

Since Q and H are both irreducible between α_1 and α_2 and have only points of these sets in common, while Q is not the union of two indecomposable continua, it follows by an extension of Rosenthal's theorem† that $Z-(Q+H)$ has two principal components S_1 and S_2. Similarly, let T_1 and T_2 be the principal components of $Z-(Q+K)$.

As K^* is connected, it lies in but one component of $Z-(Q+H)$; suppose that $K^*\cdot S_1=0$. Since $(F+Q)\cdot S_1=0$ and S_1 has frontier points on M, $R\cdot S_1\neq 0$ and $S_1\subset R$. Likewise either T_1 or T_2, say T_1, is a part of R. As $F=H+K$, every point of F is a frontier point of either S_1 or T_1, and a fortiori of R.

5. THEOREM. *Let $F=H+K$ be the union of two bounded continua and let $H\cdot K$ be the sum of a finite number n of closed sets $\{\alpha\}$ between each pair of which both H and K are irreducible. Then the number of principal components of $Z-F$ is at least n.*

PROOF. Let R_i be any component of $Z-F$ having frontier points on both H^* and K^*, let P be a bounded irreducible $S(H^*, K^*)$, and let $P_i=P\cdot R_i$. Suppose that $\overline{P}_i$ has a subcontinuum containing points of more than one set α for only k values of i, say $i=1, 2, \cdots, k$, where $k<n$. We first show that $k\geqq 1$. For, if α' denotes any α and α'' the sum of the remaining sets α, the continuum P contains a connected

† W. A. Wilson, *On the separation of the plane by irreducible continua*, this Bulletin, vol. 33, pp. 733–744, §5.

set having no points of α' or α'', but limiting points on both. This connected set must be a part of some P_i.

In some work done elsewhere† it was shown that there is a continuum Q, which is a bounded irreducible $S(H^*, K^*)$, constructed as follows: Q is the union of k arcs $\{A_i\}$, each A_i lying in R_i, $i=1, 2, \cdots, k$, and k continua $\{B_i\}$, such that $B_i \cdot B_{i'}=0$ if $i\neq i'$, each B_i is irreducible between an end of A_i and one of A_{i+1} $(i+1=1$, if $i=k)$ and meets no other points of $\sum_1^k A_i$, and $H \cdot K \subset \sum_1^k B_i \subset P$. Moreover, in the demonstration referred to, the circle used may be replaced by a polygon so large that, if I_i is its interior, no component of $\overline{P}_i-\overline{P}_i \cdot I_i$ has points on two sets α. Consequently any connected sub-set of any B_i having limiting points on two sets α, but no points on F, must lie in some R_i, $i>k$.

Since $n>k$, some B_i contains points of two or more sets α. Hence for some $i>k$, there is a connected sub-set j of B_i having limiting points on two sets α, but containing no points of F.‡ Then j lies in some P_i, where $i>k$ by the last part of the previous paragraph. Therefore $\overline{P}_i$ contains points of more than one set α, contrary to the assumption at the beginning of the proof.

Therefore, for at least n values of i, $\overline{P}_i$, and consequently $\overline{R}_i$, contains a continuum having points on more than one set α. Then §4 shows that F is the frontier of each such R_i. Hence the theorem is proved.

COROLLARY. *Let $F=H+K$ be the union of two bounded continua and for every integer n let $H \cdot K$ be the sum of n closed sets such that both H and K are irreducible between each pair. Then the number of principal components of $Z-F$ is infinite.*

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† See reference to paper by the author under §1, §§6 and 7. The hypothesis in these sections requires R to be a principal component of $Z-F$, but the demonstration only requires that R have frontier points on both H^* and K^*.

‡ Anna M. Mullikin, *Certain theorems relating to plane connected sets*, Transactions of this Society, vol. 24, Theorem 1.

6. THEOREM. *For the bounded decomposable continuum F to be the frontier of exactly n components of its complement, it is necessary and sufficient that F be the union of two continua H and K such that H · K is the sum of n, but of no finite number greater than n, closed sets between each pair of which both H and K are irreducible.*

PROOF. If F is decomposable and $Z-F$ has n principal components, it has been shown elsewhere† that F is the union of two continua H and K such that $H \cdot K$ is the sum of a finite number, greater than or equal to n, of closed sets between each pair of which both H and K are irreducible. On the other hand, if $H \cdot K$ can be expressed as the sum of n closed sets between each pair of which both H and K are irreducible, there are at least n principal components of $Z-F$ by §5.

The combination of these statements gives the theorem.

REMARKS. The decomposition of $H \cdot K$ into n closed sets given above is unique. For, if there were two different decompositions into n closed sets with the properties assigned, say $H \cdot K = \sum_1^n \alpha_i$ and $H \cdot K = \sum_1^n \beta_i$, some β_i, say β_1, would contain points of more than one set α_i. Let $\beta_1 \cdot \alpha_1 \neq 0$ and let $\beta_{11} = \beta_1 \cdot \alpha_1$ and $\beta_{12} = \beta_1 \cdot (H \cdot K - \alpha_1)$. Then $H \cdot K = \beta_{11} + \beta_{12} + \sum_2^n \beta_i$ is a decomposition into $n+1$ closed sets between each pair of which both H and K are irreducible. We then have the contradiction that $Z-F$ has at least $n+1$ principal components.

7. THEOREM. *For the bounded decomposable continuum F to be the frontier of an infinity of components of its complement, it is necessary and sufficient that F be the union of two continua H and K such that for every integer n the set H · K is the sum of n closed sets between each pair of which H and K are irreducible.*

PROOF. The necessity of these conditions was shown in

† See reference to paper by the author under §1.

the proof of the theorem referred to in §6. That they are sufficient follows from §5, Corollary.

8. *Two Examples*. It might be inferred that the statements of the theorems in §§6 and 7 are unnecessarily complicated and that, if $H \cdot K$ is the sum of an infinite set of closed sets between each pair of which both H and K are irreducible, then $Z-F$ has an infinite set of principal components. This is not in general true. The following examples show the existence of two continua H and K having these properties, but such that $Z-F$ has no principal component in Ex. I and but two principal components in Ex. II.

EXAMPLE I. Let $Q=OABC$ be a closed rectangle, such that the lengths of OA and AB are 1 and 1/2 unit, respectively. Let q be the frontier of Q. Let M be a Cantor set extending from O to A, whose complementary open intervals $\{I_n\}$ are ordered according to size.

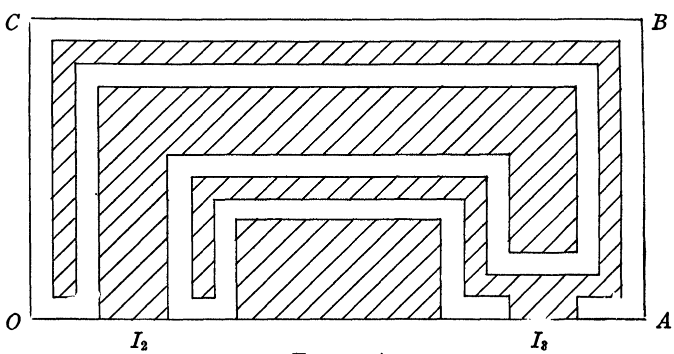

FIGURE 1

Now let a finite ordered set of closed rectangles, each of width d and of length greater than d, such that each rectangle has in common with the one preceding and the one following two different squares of side d but no point in common with any other rectangle, be called a *band*† of width d. It is

† For a complete discussion of the "method of bands" as applied to indecomposable continua see B. Knaster, *Un continu dont tout sous-continu est indécomposable*, Fundamenta Mathematicae, vol. 3, pp. 247–287. See also a memoir by K. Yoneyama, Tôhoku Mathematical Journal, vol. 12, pp. 60–62.

easily seen that there is in Q a unique band B_1 of width $1/3$, which is contiguous to all points of $q-I_1$ and whose frontier q_1 is the union of $q-I_1$ and a broken line b_1 meeting q only in the end-points of I_1. Then $Q-B_1$ consists of I_1 and a simply connected region E_1 whose frontier is I_1+b_1; let $G_1=Q-B_1$.

Likewise, in B_1 there is a unique band B_2 of width $1/3^2$ whose frontier q_2 is the union of q_1-I_1 and a broken line b_2 meeting q_1 only in the end-points of I_2. Then $G_2=B_1-B_2$ consists of I_2 and a region E_2 whose frontier is I_2+b_2. This construction can be repeated indefinitely; in the figure the shaded area is $G_1+G_2+G_3$; the unshaded with its border is B_3.

Set $H=Q-\sum_1^\infty G_n=\prod_1^\infty B_n$. Obviously H is a continuum. The Cantor set M may be regarded as the sum of an infinity of closed sets without common points. These are the set $O+A$, an enumerable set of sets each consisting of the end-points of an interval I_n, and a non-enumerable set of sets each of which is one of the other points of M. Let α be any of these sets. It is comparatively easy to show that H is irreducible between each pair of sets α.

Let K be the continuum symmetrical to H with respect to OA. Then K is also irreducible between each pair of sets α and $H\cdot K=\sum\alpha$. But each component of $Z-F$ has for its frontier a pair of continua of condensation of H and K.

EXAMPLE II. This is a variation of Ex. I. Take a closed rectangle Q whose length and width are $43/27$ units and 1 unit respectively. Let ab be one side and let the points c, d, e and f divide ab into five intervals of lengths $ac=1/3$; $cd=2/27$; $de=1$; $ef=2/27$; $fb=1/9$. Let $\{I_k\}$ be an enumerable set of open intervals divided into three classes as follows: $\{I_{3n}\}$ is the set of complementary intervals of a Cantor set M in the interval de ordered as in Ex. I; $\{I_{3n-2}\}$ and $\{I_{3n-1}\}$ are two other sets, of which the first are $I_1=cd$ and $I_2=ef$, and the others will be defined later. Let q be the frontier of Q.

For $k=1, 2, 3$, let B_k, q_k, b_k, E_k, and G_k be defined as in Ex. I. For $n=2$, let $I_k=I_{3n-2}=I_4$ be an open interval of length $1/9$ at the center of one of the longest segments of

b_1 perpendicular to, but not meeting, ab. Then B_4 is a band of width $1/3^4$ contiguous to $q_3 - I_4$. This gives q_4, b_4, E_4, and G_4. In this case, we note that $E_1 + G_4$ is a simply connected region whose frontier is $I_1 + (b_1 - I_4) + b_4$. In like manner, we let I_5 be an open interval of length 1/9 at the center of one of the longest segments of b_2 perpendicular to and not meeting ab and define B_5, q_5, b_5, E_5, and G_5. Here $E_2 + G_5$ is a simply connected region whose frontier is $I_2 + (b_2 - I_5) + b_5$.

We now return to I_6, which has been defined above. Let this process be repeated indefinitely. Each I_{3n} lies on de; each I_{3n-2} is an open interval of length 1/9 at the center of one of the longest segments of b_{3n-5} perpendicular to and not meeting ab; and similarly for I_{3n-1}. In the figure the shaded

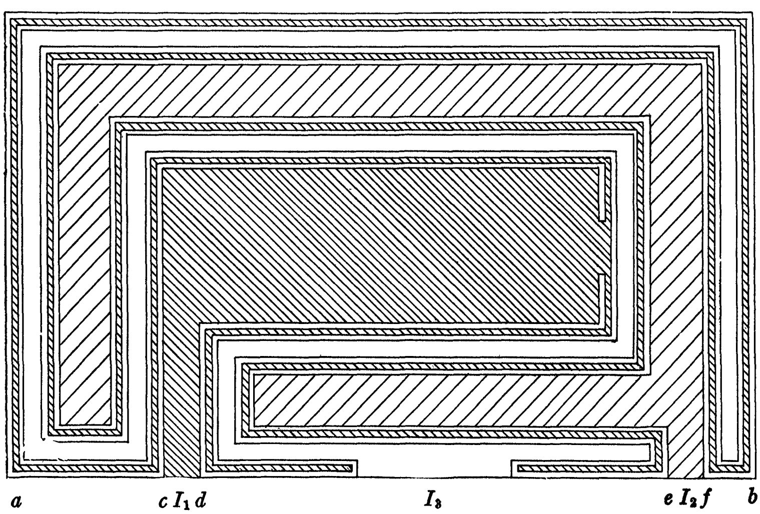

FIGURE 2

areas are $E_1 + G_4$ and E_2; the narrower white area is B_4; and the other white area is E_3. For every value of k, Q is the union of B_k and a finite number of the sets G_k. No two of the sets G_{3n} or of the corresponding regions E_{3n} have common points. For each n, $E_1 + G_4 + \cdots + G_{3n-2}$ and $E_2 + G_5 + \cdots + G_{3n-1}$ are simply connected regions whose frontiers are the union of I_1 and I_2, respectively, with a broken line interior to Q.

Let $H=Q-\sum_1^\infty G_k=\prod_1^\infty B_k$. Obviously H is a continuum. The set $ac+fb+M$ is a closed set which may be regarded as the sum of an infinity of closed sets without common points. These are the set $ac+fb$, an enumerable set of sets each consisting of the end-points of an interval I_{3n}, and a non-enumerable set of sets each of which is one of the other points of M. Let α denote any of these sets. It is comparatively easy to show that H is irreducible between each pair of sets α.

Reflect the above figure about ab and denote corresponding sets by primes. Then $K=H'$ is a continuum irreducible between each pair of sets α and $H \cdot K=\sum\alpha$. The components of $Z-F$, where $F=H+K$, are the exterior of the rectangle $Q+Q'$, the enumerable set of regions $G_{3n}+G'_{3n}$, each of which has as its frontier a pair of continua of condensation of F, and two other regions. These are $\sum_1^\infty G_{3n-2}+\sum_1^\infty G'_{3n-2}$ and $\sum_1^\infty G_{3n-1}+\sum_1^\infty G'_{3n-1}$. Each of these has the frontier F and is therefore a principal component of $Z-F$.

The essential difference between the two examples is that in Ex. I, $H \cdot K$ cannot be divided into any finite number of closed sets between each pair of which H and K are irreducible, while in Ex. II, $H \cdot K$ can be divided into precisely two, but no more, such sets, namely the sets $ac+fb$ and M. Thus Ex. II is in strict accordance with §§6 and 7.

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