SEVERI ON ALGEBRAIC GEOMETRY

Trattato di Geometria Algebrica. Volume I, Part I: *Geometria delle Serie Lineari.* By F. Severi. Bologna, 1926. viii+358 pp.

It is impossible to understand the principles on which a serious mathematical work is constructed unless one is in possession of a key to the author's thought. This should be found in the introduction, that is what an introduction is for, and in the present case Professor Severi sets forth his guiding principle in no uncertain terms.

"Con questo volume inizio la pubblicazione di un ampio Trattato, nel quale,—se il tempo e le forze mi basteranno—vorrei raccogliere, coordinare, completare dove occorra, tutto quanto vi è d'importante nel campo della geometria algebrica."*

To collect, coordinate and complete, when necessary, everything of importance in the field of algebraic geometry. This is the vast design which the author has set before himself. Consequently in the choice of subject matter and method, preference is given to that which has most connection with the general plan rather than to that which is immediately applicable to special problems.

The sub-heading of this first volume tells us that it deals with linear series, which means linear series of point groups on an algebraic curve. Why give so much attention to this particular topic? One is tempted to answer at once, "Because Professor Severi is an Italian, and this is an Italian specialty, almost a monopoly." That is no answer at all. A second answer would be because of the close connection with the study of algebraic functions and their integrals. But in the whole volume of 350 pages there is not a sign of integration, and scarcely any reference to a Riemann surface. All that will be very fully treated in subsequent volumes, but it is still far distant. What then? We must consider the group. The author in the present volume is concerned with algebraic curves, and with those properties of such curves which are unaltered by a general birational transformation. Such a transformation will carry a linear series into a linear series, it will preserve the genus of the curve and the structure of the Weierstrass points. The theory of linear series is the trunk whence all the branches of this birational geometry emanate. For instance, any non-hyperelliptic curve of genus p can be birationally transformed into a non-singular curve of order $2p-2$ in a space of $p-1$ dimensions, and this again can be carried into a plane curve if desired. Consequently in studying the birationally invariant properties of a curve, we may disregard the dimensionality of the space wherein it lies, and imagine that it is plane, or else skew and non-singular as we prefer. Herein we have a fundamental point

* P. vii.

of divergence from Professor Severi's first book which concerned itself primarily with plane curves.*

The first chapter is devoted to a study of linear systems of algebraic plane curves. This is a very large subject, but the author is concerned only with linear systems of curves as a means of reaching linear systems of point groups, and so leaves most of the topics on one side. What is essential is to establish the following theorem: An irreducible r-parameter system of algebraic curves is linear when, and only when, a single curve passes through r generic points, and the general curve is counted only once. An equal importance attaches to the beautiful theorem of Bertini: The general curve of a linear system will have a singular point only when this is a fixed singularity for all curves of the system.

Linear systems of curves lead to linear systems of point groups. Here, however, we may rise to as many dimensions as we please. Take an algebraic curve in a space of N dimensions. Cut it by a system of hypersurfaces linearly dependent on $r+s+1$ of their number, which are linearly independent, but of which s contain the curve, the result will be a linear r parameter series of point groups, a g_n^r, on our curve. We may, once for all, leave the s hypersurfaces out of our consideration. The most general algebraic curve in a space of $r+s$ dimensions may be written

$$\rho x_i = f_i(x,\ y), \qquad F(x,\ y) = 0.$$

If the system of hypersurfaces be written

$$\sum_{i=0}^{i=r} \psi_i(x_0 \cdots x_N) = 0$$

we have effectively

$$F(x,y) = 0, \quad \sum_{i=0}^{i=r} \phi_i(x,y) = 0,$$

that is to say, the intersections of a plane curve by the curves of a linear system. Conversely, if we write

$$X_i = \phi_i(x,\ y), \qquad F(x,\ y) = 0,$$

and if the series be not composite, i.e., if the generic curve ϕ through a point of F does not automatically pass through another point of F variable with the first, why then our original curve is birationally transformed to a plane curve, and this birationally to another space curve where the series is cut by the hyperplanes of the space. The problem of linear series may thus be treated either as a problem *in plano*, or else as a problem of series cut by hyperplanes. The whole art consists in the skillful alternation of these two points of view.

At this point the author encounters his first serious difficulty, to explain the meaning of a branch of a curve. He turns it with great skill. First he shows how to transform a curve birationally into a space curve which has no singular points. This may be projected down to a non-singular curve in 3-space, and this again to a plane curve whose only singularities are nodes. The complete vicinity of any point of this curve is given by one,

* *Vorlesungen über Algebraische Geometrie*, Leipzig, 1921, a translation and amplification of his previous lithographed lectures.

or at most two, power series developments for x and y in terms of an auxiliary parameter t. Hence the complete vicinity of any point of the first curve is given by a finite number of such developments. If there be ν of these for a given point we shall say that there are ν branches there. How do we know that this number ν does not depend upon the method of transformation? To begin with we may show that if two curves are birationally related the exceptional points in the transformation are singular points. Hence if two non-singular space curves are birationally related there are no exceptional points. Hence the number of branches at a singular point is the number of corresponding developments on any non-singular space curve.

I must confess to being struck with admiration at the author's ingenuity in treating the problem of branches in this fashion. A better treatment for his purpose could not be devised. But the treatment, like all things human, has the defects of its qualities. No geometer has reached the heart of the singular-point difficulty, till he has made a thorough study of the power series developments.

Suppose that we have a linear series on a curve, a g_n^r, an r-parameter series of groups of n points. Certainly $n \geqq r$, for there can not be more parameters than points. Is it possible that our series is contained in a $g_n^{r'}$, where $r' > r$ and this in a $g_n^{r''}$, where $r'' > r$, etc.? Yes, this is certainly possible, but we must come to an end some time. Very well, when we have a series which is not contained in another of the same order but larger dimension, it is said to be "complete." How do we know that a given series is not contained in two different complete series? The usual proof is by the classical residuation theorem, the "Restsatz" of Brill and Nöther. But Severi, at this point, has none of the necessary apparatus at his disposal. He therefore turns to arithmetical and analytical considerations. Two groups of the same number of points are said to be "equivalent" if they are groups of the same series. Sums and differences of groups are defined and the laws of addition and subtraction established. If two series have a common group they are both contained in a larger series. Hence there can be but one complete series containing a given series, for if there were two, both would be contained in a third larger. I will hazard the guess that Severi must have regretted the very slight introduction of the zeros and poles of an algebraic function that he needed to prove this theorem, as a breach in the uniformity of his method; it was a very small breach at most.

In Chapter IV we reach the Jacobian group of a series, the group of multiple points, and a proof that this varies continuously with the series. This leads to the strange theorem, which requires simple transcendental considerations, that if we have two one-dimensional series, the difference between the Jacobian group and twice a general group of one, is equivalent to the corresponding difference in the other. There exists thus a single birationally invariant series which includes every such difference. Let the order of this invariant series be $2p-2$. Then p is defined as the "genus" of the curve and easily proved to be a positive integer when not zero. An immediate application is made in finding the group of points of multiplicity $r+1$ of a g_n^r, Brill's formula.

We have at last all the essentials for the theory of linear groups, order, dimension, completeness, deficiency, canonical series, all present, all deduced by very rapid and simple operations. Anyone who is familiar with the subject will be struck by the beauty of it all. How will it impress the beginner? Perhaps he will feel the same way about it. But I cannot escape the feeling that he may find difficulty in understanding the motivation of the different steps. Why does the juggler take a hat from the table? Because he wishes to pull a rabbit out of it? But why a rabbit? Because well-born rabbits come out of hats? If the reader has this feeling at all, it will be intensified in the next chapter, the fifth, which is called the geometry of linear series, "secondo il metodo rapido." It certainly is rapid. Given an algebraic plane curve which is irreducible. Let $p+1$ be the smallest number of generic points such that the complete series including their group has not the dimension 0. Then p is defined as the genus of the curve and is shown to be zero when, and only when, the curve is rational. If the order of a complete generic series be $>p$, then $n-r=p$. If $n-r<p$, the series is said to be "special." These facts being birationally invariant, we may confine ourselves to the consideration of curves with no singularities but nodes. We then introduce the idea of adjoint curves, residual groups, and so the Riemann-Roch theorem and its consequences by the classical processes of Brill and Nöther. It is beautifully done and in a chapter of 25 pages the reader becomes acquainted with all the important theorems about linear series. But if one has learned elsewhere to associate the genus of a curve with a shortage of singular points, this new definition seems to be dragged in by the heels. Still, the result is beautiful.

In Chapter VI, the author turns to the theory of algebraic correspondence between curves and upon one curve. This is the longest and, it seems to me, the most important chapter in the book. There is a vast amount of literature upon the subject, the names of Chasles, Cayley, Brill and Zeuthen come at once to mind. No one can get to the very bottom of this theory without the use of abelian integrals, but the introduction through linear series which Severi exhibited in previous writings and which he gives here in full, is very far ahead of any other with which I am familiar.

The chapter begins with a study of birational transformations of a curve into itself, which leads to the study of projective transformations of a curve of higher space into itself. These are finite in number when the genus is above unity. We thus get the idea of moduli, or birationally invariant numbers connected with a curve. More generally, suppose that we have an algebraic transformation of a plane curve into itself. This may always be written

$$f(x, y)=f(x', y')=0,$$
$$\phi_1(x, y, x', y')=\phi_2(x, y, x', y')=0.$$

Castelnuovo determined an upper limit for the number of coincidences of such a correspondence and showed that it is reached when, and only when, there is need of but one auxiliary equation, $\phi_1=0$.

Here the groups of points (x', y') corresponding to a variable (x, y) are, with (x, y) itself counted γ times, the groups of a linear series. We call $\gamma<0$ the *value* of the correspondence. When positive it is an integer. On the other hand, if P correspond to $P_1 \cdots P_\nu$ and Q to $Q_1 \cdots Q_\nu$ and if $(-\gamma)$ be such a positive integer that the groups $P_1+P_2+ \cdots +P_\nu+ (-\gamma)Q$ and $Q_1+Q_2+ \cdots +Q_\nu+(-\gamma)P$ are equivalent, we say that the correspondence has the negative integral value γ.

It may be shown by transcendental methods which Severi does not reproduce that on a curve of general moduli there are no correspondences but those of integral or zero value.

The value of the sum of two correspondences is the sum of their values, the value of the product, i.e., of the succession of two correspondences, is the negative of the product of their values. A pencil of curves will cut a correspondence of value unity. Hence we may construct a correspondence of any positive or negative integral value.

If we have a ν to ν' correspondence of value 0, the number of coincidences, which arise from the coalescence of two points on the same branch, is $\nu+\nu'$. Severi* proves this with great care, or rather proves the broader theorem that the sum of the groups corresponding to a given point in such a correspondence and its inverse is equivalent to the group of self-corresponding points. When it comes to correspondences of positive or negative value, we have merely to piece them out to correspondences of value zero by adding known correspondences of opposite value. We thus reach the Chasles-Cayley-Brill correspondence value

$$\nu+\nu'+2p\gamma.$$

In reaching this number, great care must be taken to count multiple coincidences correctly. The rule was found by Zeuthen and is carefully explained by Severi†; he also shows that this number is birationally invariant.

The value of a correspondence leads naturally to the important idea of the linear dependence of correspondences. The chapter closes with the discussion of another characteristic number of a correspondence called its "grade" which depends upon the coincidences between the correspondence and another differing infinitesimally from its inverse. I am willing to take the author's word that this is important, but must confess to finding it a bit obscure.

The great theory of the algebraic correspondence of curves developed in Chapter VI was based on the theory of linear series. The reader would naturally expect further developments in the subsequent chapters. Not at all; the author goes back to his original theme and exhibits two other methods of building up the theory "ab initio." In Chapter VII this is done by the methods of projective geometry in higher space, first brought to perfection by Segre and Castelnuovo, or Castelnuovo and Segre; it would be hard to settle the question of priority. The first thing is to find the complete series containing a given series. The essential point is to show that if

* Pp. 221 ff.

† Pp. 224 ff.

two linear series have a common group, they are contained in a series of higher dimension. The plan consists in studying groups on curves in spaces of r and r' dimensions and the ruled variety in $r+r'+1$ dimensions generated by the lines connecting the two, thus getting eventually a more ample system of spaces of $r+r'$ dimensions than that originally given. The next step consists in proving once more the Riemann-Roch theorem, and this is done by means of a formula already established for the number of groups of $r+1$ points common to a g_n^r and a g_m^1. The chapter is short and sketchy.

In the final chapter, Severi develops the theory of linear series by what he calls the "metodo algebrico." He takes the various steps in the following order:

1. A short account of Cremona transformations.
2. Study of the particular case of quadratic transformations.
3. A discussion of Nöther's idea of clusters of infinitely near singularities and a demonstration that any singular point can be treated as such a cluster.
4. A proof that the first polar of a general point is an adjoint curve whether the singularities are distinct or clustering. This shows that the number of clustering singularities must be finite. The proof that the form of the cluster is independent of the type of transformation used cannot be given satisfactorily without a study of the exponents in the power series developments.
5. A rather laborious proof that the known singularities of a curve impose independent conditions upon another curve of sufficiently high order.
6. Nöther's $AF+B\phi$ theorem.
7. Definition of adjoint curves, and residue theorem. Demonstration that the adjunction conditions are independent.
8. Riemann-Roch theorem as before.

It is part of a reviewer's duty to put the reader on guard against the writer's predilections and prejudices; a candid reviewer should also give warning against his own personal preferences. And the present reviewer must confess to a feeling of regret that the book was not written backwards, with this algebraic method explained at the outset, and the geometric methods added subsequently. This classical procedure which has been followed by Brill and Nöther and Bertini seems to me to penetrate more completely into the heart of the fundamental questions involved, especially when joined to a study of the series developments. Such a study will surely be made in subsequent volumes. The author sees the whole edifice which he plans to build; the reviewer can only see in part, and so he must make the mistake of judging the part separated from the whole. When shall we see the whole? That secret is still in the lap of the gods. The author says in the preface: "Con questo volume inizio la publicazione di un Trattato—se il tempo e le forze mi basteranno. . . . . " Every reader will hope ardently that they do.

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