EXPANSION IN FACTORIALS

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An expansion of the form
\[ f(z) = \sum \frac{b_n}{(z + 1)(z + 2) \cdots (z + n)} \]
can be obtained from the consideration of Cauchy's formula
\[ 2\pi i f(z) = \int_C \frac{f(\xi)d\xi}{\xi - z}, \]
if \( f(\xi) = 0 \) at infinity, together with the result†
\[ \frac{n!}{(z + 1)(z + 2) \cdots (z + n + 1)} = \int_0^1 u^n (1 - u)^{z}du, \]
where \((1 - u)^z\) denotes the branch reducing to unity for \( u = 0 \). The above relations can also be used for deriving an expansion in series of non-inverted factorials. By (1) we have
\[ \frac{1}{z - t} = \int_0^1 (1 - u)^{z-t-1}du. \]
Consider
\[ (1 - u)^{z-t-1} = (1 - u)^{s-1}(1 - u)^{-t-1}. \]
Since
\[ (1 - u)^z = 1 \]
when \( u = 0 \), we may write
\[ (1-u)^z = 1 - \frac{z}{1!} u + \frac{z(z-1)}{2!} u^2 - \cdots + \frac{(-1)^n}{n!} z(z-1) \cdots (z-n+1) u^n + \cdots. \]

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The binomial expansion (2) will be uniformly convergent for \(0 \leq u \leq 1\) when \(R(z) > 0\).* Also let us introduce the condition
\[ R(-l-1) > 0, \]
that is,
\[ R(l) < -1. \]
Then the expansion
\[ (1 - u)^{l-t-1} = (1 - u)^{-t-1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} u^n(1 - u)^{-t-1}z(z - 1) \cdots (z - n + 1), \]
which holds for \(0 \leq u \leq 1\), can be integrated termwise, so that we may write
\[ \frac{1}{z-t} = \int_0^1 (1 - u)^{l-t-1}du = \int_0^1 (1 - u)^{-t-1}du + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left[ \int_0^1 u^n(1 - u)^{-t-1}du \right] \cdot z(z - 1) \cdots (z - n + 1). \]
But
\[ \int_0^1 u^n(1 - u)^{-t-1}du = \frac{-n!}{t(1-t)(2-t) \cdots (n-t)}, \]
and hence we have
\[ \frac{1}{z-t} = -\frac{1}{t} - \sum_{n=1}^{\infty} \frac{(-1)^n z(z-1)(z-2) \cdots (z-n+1)}{t(1-t)(2-t) \cdots (n-t)}, \]
where \(R(z) > 0, R(t) < -1\).

Let \(R_n(u)\) denote the remainder after \((n+1)\) terms of the series \((2)\) multiplied by \((1-u)^{-t-1}\). Since \((2)\) is uniformly convergent, given \(\epsilon, n_0\) can be found so that for \(n \geq n_0\) and all \(u, 0 \leq u \leq 1\), we have
\[ |R_n(u)| < \epsilon. \]

* \(R(z)\) denotes the real part of \(z\).
If we let \( R_n'(t) \) denote the remainder after \((n+1)\) terms of the series (5), we may observe that
\[
R_n'(t) = \int_0^1 R_n(u) \, du,
\]
and hence
\[
|R_n'(t)| \leq \int_0^1 |R_n(u)| \, du < \epsilon,
\]
where \( \epsilon \) is independent of \( t \); consequently, (5) is a uniformly convergent series in \( t \).

Let \( f(z) \) be a function analytic on and outside a closed contour \( C \) situated to the left of \( R(z) = -1 \), and vanishing at infinity; then
\[
-2\pi if(z) = -\int_C \frac{f(t) \, dt}{z - t}
\]
\[
= \int_C f(t) \left[ \frac{1}{t} + \sum_{n=1}^{\infty} \frac{(-1)^n (z-1) \cdots (z-n+1)}{t(1-t) \cdots (n-t)} \right] \, dt
\]
when \( R(z) > 0 \). Since integration termwise is justifiable, we have
\[
-2\pi if(z) = \int_C \frac{f(t) \, dt}{t} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(1-t) \cdots (n-t)} z(z-1) \cdots (z-n+1).
\]
Hence we may state the following theorem.

**Theorem I.** Let \( f(z) \) be a function analytic on and outside of a closed contour \( C \) situated to the left of \( R(z) = -1 \), and vanishing at infinity; then for all \( z \) with \( R(z) > 0 \)
\[
f(z) = b_0 + b_1 z + b_2 z(z-1) + \cdots + b_n z(z-1) \cdots (z-n+1) + \cdots,
\]
where
\[
b_n = \frac{(-1)^{n+1}}{2\pi i} \int_C \frac{f(t) \, dt}{t(1-t) \cdots (n-t)}.
\]
If we take \((1-u)^{t-1}\) as
\[(1-u)^{s+k} \cdot (1-u)^{-t-1-k},\]
where \(k\) may be complex, repeat the steps by means of which Theorem I was deduced, and replace \(z\) by \(z+k\) and \(t\) by \(t+k\), we find the following generalized theorem.

**Theorem II.** Let \(f(z)\) be a function analytic on and outside of a closed contour \(C\) situated to the left of \(R(z) = -R(1+k)\), and vanishing at infinity, then for all \(z\) with \(R(z) > -R(k)\), we have

\[
f(z) = b_0 + b_1(z + k) + b_2(z + k)(z + k - 1) + \cdots + b_n(z + k)(z + k - 1) \cdots (z + k - n + 1) + \cdots,
\]

where

\[
b_n = \frac{(-1)^{n+1}}{2\pi i} \int_C \frac{f(t)dt}{(t+k)(1-t-k)(2-t-k) \cdots (n-t-k)}.
\]

Let \(U\) be max. \(|f(t)|\) on \(C\) and \(l\) the length of \(C\); then considering the expansion defined by Theorem I, it is observed that \(t = t_1 + it_2\) has \(-t_2 > 1\), since \(R(t) < -1\) so that \(|n-t| \geq n-t_2 > n+1\), and hence

\[
\frac{1}{|t(1-t) \cdots (n-t)|} < \frac{1}{(n+1)!}.
\]

Consequently

\[
|b_n| < \frac{h}{(n+1)!} \quad \left( h = \frac{U}{2\pi} \right).
\]