AN ELEMENTARY PROPERTY OF BOUNDED DOMAINS*

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It is the purpose of this note to prove the following property of a bounded domain‡ lying in a two-dimensional euclidean space.

THEOREM. *If P is a point of a plane and bounded domain, there exists a triangle such that it contains P in its interior, has its vertices on the boundary of the domain, and every other point of the triangle and its interior lie in the domain.*

PROOF. Let D be the domain and B denote the boundary of D. Let r denote the lower limit of all numbers $d(P, x)$, where x is any point of the boundary B and $d(P, x)$ denotes the distance from P to x. Let C be the circle with center at P and radius r. Since B is a closed set, the circle C contains a point X of B. Let E_1 and E_2 be points whose distance from P is $\frac{1}{2}r$ and which lie on the line through P perpendicular to the line XP. Let $E_iF_i(i=1, 2)$ be a ray through E_i parallel to the ray XP.§ If y is any point of the ray XP such that $d(P, y) \geqq r$, let C_y be the circle with center at P and

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‡ A connected set of points D is said to be a *domain* if for every point P of D there is a circle with center at P, such that D contains every point in the interior of the circle. The *boundary* of a domain is the set of all limit points of the domain which do not belong to the domain. The boundary of a bounded domain is a closed and bounded point set, and if H is any connected point set containing a point of the domain and a point not belonging to the domain, then H contains a point of the boundary of the domain.

§ If l is a line and A and B are points of l, the *ray* AB (but not the ray BA) is the maximal connected subset of $l-A$ which contains B. If AB is a ray and C is a point not on the line AB then if l denotes the line through C parallel to the line AB the *ray through* C *parallel to the ray* AB is the maximal connected subset of $l-C$ which lies on the B-side of the line AC.

radius $d(P, y)$ and let y_i be the intersection of C_y and the ray E_iF_i. Let $[y]$ be the set of all points y of the ray XP such that $d(P, y) \geqq r$ and the arc y_1yy_2 of C_y contains a point of the boundary B. Using the fact that the boundary B is closed and the ray XP contains at least one point of B, it is not difficult to show that the set $[y]$ is non-vacuous and closed. Since $[y]$ is closed, there is a first point A of $[y]$ on the ray XP.* The arc A_1AA_2 of C_A contains a point Y of the boundary B and every point of the segment XY belongs to the domain D. There are two cases to consider according as $Y=A$ or $Y\neq A$.

Case I. If $Y\neq A$, let Q be the intersection of the line XY and the line through P parallel to the line AY. Let $[z]$ be the set of all points z of the ray QP such that either (a) the segment Xz contains a point of the boundary B, or (b) the segment Yz contains a point of B, or (c) the point z belongs to B. It follows easily that the set $[z]$ is non-vacuous, closed and contains no point of the interval QP. Let G be the first point of the set $[z]$ on the ray QP. Then either G is a point of the boundary B or the segment XG or YG contains a point of the boundary B.

Since G does not lie on the interval QP, the point P is in the interior of the triangle XYG, and every point in the interior of this triangle lies in the domain D. If the segment XG contains a point of B, let Z be the first point of B on the ray XG. We may show that there is a first point of B on the ray XG as follows. Since the ray XG makes an acute angle with the ray XP, the ray XG contains a point H of the circle C. Now the points of B on the ray XG form a set which is closed except possibly for the point X. But X is not a limit point of the set of points of B which lie on the ray XG, since no point of the segment XH belongs to B. Hence the points of B which lie on the ray XG form a closed set and there is a first point Z of B on the segment XG in the order X to G.

* If K is a set of points lying on a ray AB, a point p of K is said to be the *first point of* K on the ray AB if for every point q of K distinct from p the point p lies on the segment Aq.

Let l be the line through X parallel to the line AY and let K denote the point of intersection of l and the circle C which is distinct from X (the line l is not tangent to the circle C since $A \neq Y$). The segment YK contains the point P and there is no point of B on or inside the triangle XYK except possibly its vertices. Hence the angle XYZ is greater than the angle XYK and so P lies in the angle XYZ. Similarly the angle YXZ is greater than the angle YXA and so P lies in the angle YXZ. Then P lies in the interior of the triangle XYZ. The segment XZ belongs to the domain D, since the segment XH belongs to the domain D and Z is the first point of B on the ray XG. The segment YZ and the interior of the triangle XYZ belong to the domain D, since they are subsets of the interior of the triangle XYG. Therefore the triangle XYZ satisfies all the conditions of our theorem and is the desired triangle.

If no point of the segment XG belongs to the boundary B, but the segment YG contains at least one point of B, then if Z is chosen as the first point of B on the ray YG, the same proof shows that the triangle XYZ is the desired triangle. If neither the segment XG nor the segment YG contains a point of B, the triangle XYG satisfies all the conditions of our theorem and is the required triangle.

Case II. If $A = Y$, let l be the line through P perpendicular to the line XY and let G_1 and G_2 be points of l such that P is between G_1 and G_2. On the ray PG_1, let $[z]$ denote the set of all points z such that z belongs to the boundary B or the segment Xz contains a point of B or the segment Yz contains a point of B. It is easily seen that $[z]$ is non-vacuous and closed. Let H be the first point of the set $[z]$ on the ray PG_1. The set of points of B which lie on the ray XH is closed except possibly for the point X. Since the angle YXH is acute, there is a segment XK which lies entirely in the interior of the circle C and thus entirely in the domain D. Then the set of points of B on the ray XH is closed. If the segment XH contains a point of B, let Z be the first point of B on the segment XH in the order X to H. If the segment XH contains no point of B but the segment YH contains a point of B, let Z be the first point of B on the segment YH in the order Y to H. There is a first point since the set of points of B which lie on the ray YH is closed. If neither the segment XH nor the segment YH contains a point of the boundary B, let H be the point Z. In any case the triangle XYZ has its vertices on the boundary B and every other point of the triangle and its interior belongs to the domain D, the segment XY contains the point P, and the angles XYZ and YXZ are acute.

In exactly the same manner, we determine a point W of the boundary B on the G_2-side of the line XY, such that the triangle XYW has the same properties as the triangle XYZ, where Z is replaced by W.

Since the angles XYZ, YXZ, XYW and YXW are all acute, every point of the segment ZW lies in the interior of the quadrilateral $XZYW$. If the segment ZW does not contain the point P, the point P lies in the interior of the triangle XZW or YZW. In this case the triangle XWZ or the triangle YWZ (whichever contains P in its interior) is the desired triangle. If the segment WZ contains the point P, we have a quadrilateral $XZYW$ having its vertices on the boundary B and every other point of the quadrilateral and its interior belonging to the domain D, and such that P is the intersection of the diagonals of $XZYW$. Either the rays XZ and WY have no point in common or the rays ZX and YW have no point in common. Let us suppose that the rays XZ and WY have no point in common. If the lines XZ and WY are parallel, let PQ be the ray through P parallel to the ray XZ. If the lines XZ and WY intersect in a point M, let Q be a point of the line PM such that P is between Q and M. On the ray PQ, let $[v]$ be the set of all points v such that v is a point of B or the segment Xv contains a point of B or the segment Wv contains a point of B. As above, we may show that the set $[v]$ is non-vacuous and closed. Let N be the first point of the set $[v]$ on the ray PQ. If the segment XN contains a point of B, let V be the first point of the boundary B on the ray XN. There is a first point V since the set of points of B which lie on the ray XN is a closed set. Since V is outside the quadrilateral $XZYW$, the angle XWV is greater than the angle XWZ and the angle WXV is greater than the angle WXY. Hence the point P lies in the interior of the triangle WXV and this is the desired triangle. If the segment XN contains no point of B but the segment YN contains a point of B, let V be the first point of B on the ray XN, and again the triangle XWV is the desired triangle. If neither the segment XN nor the segment YN contains a point of B, the point N belongs to B and the triangle XWN is the desired triangle. The case in which the rays ZX and YW have no point in common is exactly the same as the above.

Thus, in any possibility, we have established the existence of a triangle having the properties required in our theorem.

It is of interest to notice that while for any point of any bounded domain there is a triangle having the properties required in our theorem, there exist bounded domains, and in fact bounded domains with simple closed curves as boundaries, such that there is *no simple polygon of more than three sides* having its vertices on the boundary of the domain and so that every other point of the polygon and its interior lie in the domain. An example of such a domain is the bounded domain whose boundary is a three-cusped hypocycloid.

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