ON CONTINUOUS CURVES IN $n$ DIMENSIONS*

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If $M_1$ and $M_2$ are subsets of a connected point set $M$, the subset $K$ of $M$ is said to separate $M_1$ and $M_2$ in $M$ if $M-K$ is the sum of two mutually separated sets containing $M_1$ and $M_2$ respectively. R. L. Moore‡ has shown that in order that a plane continuum $M$ be a continuous curve§ it is necessary and sufficient that for every two distinct points $A$ and $B$ of $M$ there should exist a subset of $M$ which consists of a finite number of continua and which separates $A$ and $B$ in $M$. Consider the following example: Let $S_i$ ($i=1, 2$) be the set of all points $(x, y, z)$ in three dimensions such that $x = (-1)^i$, $-1 \leq x \leq 1$, $0 \leq z \leq 1$. Let $R_0$ be the set of all points $(x, y, z)$ such that $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, $z = 0$. For each integer $n>0$, let $R_n$ be the set of all points $(x, y, z)$ such that $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, $z = 1/n$. Let

$$M = S_1 + S_2 + \sum_{n=0}^{\infty} R_n.$$ 

It is easy to see that every two points of $M$ may be separated by a single subcontinuum of $M$ and yet $M$ is not a continuous curve. Hence the condition given by Moore is not sufficient in order that a continuum in $n$ dimensions ($n > 2$) be a continuous curve. In this paper we give two modifications (Theorems 2 and 4) of Moore's theorem which hold in $n$ dimensions.

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* Presented to the Society, October 29, 1927.
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§ We shall use the term continuous curve in the sense of a point set which is closed, connected and connected im kleinen. See R. L. Moore, Concerning simple continuous curves, Transactions of this Society, vol. 21 (1920), p. 347.
THEOREM 1.* If \( M \) is a continuous curve in euclidean space of \( n \) dimensions, \( K \) is a bounded subcontinuum of \( M \) and \( \varepsilon \) is any positive number, then there exists a set \( L \) such that

1. \( K + L \) is a continuous curve which is a subset of \( M \),
2. every point of \( L \) is within a distance \( \varepsilon \) of some point of \( K \),
3. \( L \) consists of a countable set of arcs of \( M \), not more than a finite number of which are of diameter greater than any given positive number,
4. \( L + K \) is non-dense at every point except those points at which \( K \) fails to be non-dense.

PROOF. The set \( M \) is uniformly connected im kleinen over the set \( K \).† Let \( \delta_1, \delta_2, \delta_3, \ldots \) be a sequence of positive numbers such that every two points of \( K \) whose distance from one another is less than \( \delta_m \) can be joined by an arc of \( M \) whose diameter is less than \( \varepsilon/2m \). For each point \( p \) of \( K \) and each positive integer \( n \), let \( C_{np} \) and \( C'_{np} \) be hyperspheres with center \( p \) and radii \( \varepsilon/n \) and \( \varepsilon/(2n) \) respectively.‡ By the Borel theorem, for each value of \( n \) there is a finite subset of the set \([C_{np}]\),

\[
C'_{np1}, C'_{np2}, C'_{np3}, \ldots, C'_{npn'},
\]
such that every point of \( K \) is in the interior of one of the sets \( C'_{np_i} \) for \( 1 \leq i \leq n' \). Since \( M \) is a continuous curve there are but a finite number,

\[
M_{ni1}, M_{ni2}, M_{ni3}, \ldots, M_{nim_n},
\]
of the components§ of \( M \cdot I(C_{np_i}) \) that contain points in the interior of \( C'_{np_i} \). For each \( n, i \) and \( j \), let \([K_{ni}]\) be the set of

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* This theorem contains as a special case a theorem due to H. M. Gehman, *Concerning the subsets of a plane continuous curve*, Annals of Mathematics, vol. 27 (1925), pp. 29–46, Theorem 3.
‡ If \( p \) is a point and \( r \) a positive number, the hypersphere with center \( p \) and radius \( r \) is the set of all points of the space whose distance from the point \( p \) is \( r \). If \( S \) is a hypersphere, \( I(S) \) denotes the interior of \( S \).
§ A connected subset of a point set \( H \) which is not a proper subset of any connected subset of \( H \) is called a component of \( H \).
components of $K \cdot M_{nij} \cdot I(C_{npn})$. By the Zermelo postulate*, there exists a set of points $[P_{nij}]$ such that each set $K_{nij}$ contains just one point $P_{nij}$ and each point $P_{nij}$ belongs to just one component $K_{nij}$. In the set $[P_{nij}]$ there is a finite subset,

$$P_{nij}^1, P_{nij}^2, P_{nij}^3, \ldots, P_{nij}^{k_1}, \ldots$$

such that every point of $[P_{nij}]$ is within a distance $\delta_i$ of some point of this finite set. There exists an arc $\alpha_{nij} (1 \leq r \leq k_1 - 1)$ with end points $P_{nij}^r$ and $P_{nij}^{r+1}$ and lying wholly in $M_{nij}$. There exists a finite subset,

$$P_{nij}^{k_1+1}, P_{nij}^{k_1+2}, \ldots, P_{nij}^{k_2}, \ldots$$

of the set $[P_{nij}]$ such that every point of $[P_{nij}]$ is within a distance $\delta_2$ of some point of $P_{nij}^1, P_{nij}^2, \ldots, P_{nij}^{k_1}$. Let $\alpha_{nij}^r (k_1 \leq r \leq k_2 - 1)$ be an arc of $M_{nij}$ with end points $P_{nij}^r$ and some point of $\sum_{t=1}^{k_t} P_{nij}^t$. Continue this process indefinitely except that for $t > n$ we place the additional condition on $\alpha_{nij}^t (k_t \leq r \leq k_{t+1} - 1)$ that it be of diameter less than $\epsilon/(2t)$. This can be done since any two points of $K$ whose distance from one another is less than $\delta_i$ can be joined by an arc of $M$ whose diameter is less than $\epsilon/(2t)$.

For each $n$, $i$ and $j$, there is a countable set of arcs of $M$, $\alpha_{nij}^1, \alpha_{nij}^2, \alpha_{nij}^3, \ldots$, such that (a) each lies interior to a hypersphere of radius $\epsilon/n$ with a point of $K$ as center, (b) only a finite number are of diameter greater than a given positive number, and (c) each has its end points on $K$. For each value of $n$ the numbers $i$ and $j$ range over finite sets of values; hence the set of all arcs $[\alpha_{nij}^r]$ for a fixed value of $n$ satisfy conditions (a), (b), and (c) above. And since all arcs $[\alpha_{nij}^r]$ for a fixed value of $n$ are of diameter less than $2\epsilon/n$, the set of all arcs $[\alpha_{nij}^r]$ for all values of $n$ satisfies the condi-

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† The symbol $k_1$ denotes a positive integer whose value depends on $n$, $i$ and $j$. 

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tion that only a finite number are of diameter greater than a given positive number. Let

\[ L = \sum_{n \leq m_n} a_{nij}, \quad 1 \leq i \leq n', \quad 1 \leq j \leq m_n, \quad 1 \leq r < \infty, \quad 1 \leq n < \infty. \]

We have shown that \( L \) satisfies conditions (2) and (3) of our theorem. It remains to prove that (1) and (4) are satisfied. Since only a finite number of the arcs of \( L \) are of diameter greater than a given positive number and each has a point on the closed set \( K \), every limit point of \( L \) which does not belong to \( L \) belongs to \( K \). Thus \( K+L \) is closed. Let \( P \) be any point of \( K+L \). If \( P \) does not belong to \( K \) it is easy to see that \( K+L \) is connected im kleinen at \( P \), for the interiors of hyperspheres of sufficiently small radii and center \( P \) contain no point of \( K \) and points of only a finite number of arcs of \( L \). If \( P \) is a point of \( K \) and \( \eta \) is any positive number, there is a hypersphere \( C_{nPn} \) which lies entirely in the interior of the hypersphere with radius \( \eta/4 \) and center \( P \) and such that \( I(C_{nPn}) \) contains \( P \). Let \( M_{nij} \) be the component of \( M \cdot I(C_{nPn}) \) containing \( P \). There exists a positive number \( \gamma \) such that every point of \( K \) whose distance from \( P \) is less than \( \gamma \) lies in \( M_{nij} \). There exists a number \( \rho > 0 \) such that every point \( p' \) of \( L \) whose distance from \( P \) is less than \( \rho \) lies on an arc \( \alpha_{p'} \) of \( L \), one of whose points \( e \) belongs to \( K \cdot M_{nij} \) and such that the subarc \( p'e \) of \( \alpha_{p'} \) is of diameter less than \( \eta/2 \). Let \( \sigma \) be the smaller of \( \gamma \) and \( \rho \). Now let \( Q \) be any point of \( K+L \) whose distance from \( P \) is less than \( \sigma \). If \( Q \) belongs to \( K \) it belongs to \( M_{nij} \). By the method of

* If \( S_\gamma \) and \( S_d \) denote hyperspheres with center \( P \) and radii \( \gamma \) and \( d \) respectively, then only a finite number of arcs of \( L \) have points in \( I(S_d) \) for any \( d < \gamma \) and contain no point of \( I(S_\gamma) \cdot K \) since any such arc is at least of diameter \( \gamma - d \). There is a number \( d_1 > 0 \) such that for \( d \leq d_1 \) there is no such arc. Also there is a number \( d_2 > 0 \) such that no arc of \( L \) of diameter greater than \( \eta/2 \) contains a point whose distance from \( P \) is less than \( d_2 \) unless the arc contains \( P \). On each of the finite set of arcs of \( L \) of diameter greater than \( \eta/2 \) that contain \( P \) there is a point \( q \) such that the subarc \( qP \) of the arc is of diameter less than \( \eta/2 \). Let \( d_3 \) be the smallest of the finite set of distances from \( P \) to the points \( q \). Let \( \rho \) be the smallest of the numbers \( d_1, d_2 \) and \( d_3 \).
construction of \( L \), there is a subset \( L' \) of \( L \) such that \( M_{nij} \) contains \( L' \) and \( L' + K \cdot M_{nij} \) is connected. But every point of \( L' + K \cdot M_{nij} \) is at a distance from \( P \) less than \( \eta/2 \) and \( L' + K \cdot M_{nij} \) contains both \( P \) and \( Q \). If \( Q \) is not a point of \( K \), it lies on an arc \( \alpha Q \) of \( L \) which contains a point \( e \) of \( K \cdot M_{nij} \) such that the subarc \( eQ \) of \( \alpha Q \) is of diameter less than \( \eta/2 \). Then \( \alpha Q + L' + K \cdot M_{nij} \) is a connected subset of \( L + K \) containing \( P \) and \( Q \) and such that every point is at a distance from \( P \) less than \( \eta \). Therefore \( K + L \) is connected at every point \( P \).

Let \( P \) be any point of \( K \) at which \( K \) is non-dense. Then if \( S_1 \) is any hypersphere with center \( P \), the set \( I(S_1) \) contains a hypersphere \( S_2 \) such that \( S_2 + I(S_2) \) contains no point of \( K \). Since only a finite number of the arcs of \( L \) are of diameter greater than a given positive number, there are only a finite number of arcs of \( L \) that have points in \( I(S_2) \). Then there is a hypersphere \( S_3 \) lying in \( I(S_2) \) such that \( I(S_3) \) contains no point of \( L \). Then the interior of every hypersphere \( S_1 \) with center at \( P \) contains a hypersphere \( S_3 \) such that \( I(S_3) \) contains no point of \( K + L \). Hence \( K + L \) is non-dense at the point \( P \).

**Theorem 2.** In order that a continuum \( M \) lying in euclidean space of \( n \) dimensions be a continuous curve it is necessary and sufficient that for every two distinct points \( A \) and \( B \) of \( M \) there should exist a subset of \( M \) which consists of a finite number of continuous curves and which separates \( A \) and \( B \) in \( M \).

**Proof.** The condition is necessary. Let \( d \) be the distance from \( A \) to \( B \). Let \( S_1 \) and \( S_2 \) be hyperspheres with center \( A \) and radii \( d/2 \) and \( d/4 \) respectively. Let \( H = S_1 + I(S_1) - I(S_2) \). The set \( M \cdot H \) is closed and it is easy to see that there is at least one component of \( M \cdot H \) containing points on both \( S_1 \) and \( S_2 \). As \( M \) is a continuous curve there cannot be more than a finite number of such components. Let \( K_1, K_2, K_3, \ldots, K_m \) denote the set of all components of \( M \cdot H \) which contain a point on \( S_1 \) and a point on \( S_2 \). By Theorem 1, for each \( i, 1 \leq i \leq m \), there is a continuous curve \( M_i \) which contains
$K_i$, is a subset of $M$ and such that every point of $M_i$ is within a distance $d/8$ of some point of $K_i$. Suppose that $A$ and $B$ lie in a connected subset of $M - \sum_{i=1}^{m} M_i$. Then there is an arc with end points $A$ and $B$ lying in $M - \sum_{i=1}^{m} M_i$.\(^*\) This arc contains a subarc $\alpha$ which is a subset of $H$ and has one end point on $S_1$ and the other on $S_2$. Then $\alpha$ must belong to some set $K_i$ and thus to $\sum_{i=1}^{m} M_i$. But this is impossible, for $M - \sum_{i=1}^{m} M_i$ contains $\alpha$. Therefore $\sum_{i=1}^{m} M_i$ separates $A$ and $B$ in $M$.

The condition is sufficient. If $M$ is not a continuous curve there exist two concentric hyperspheres $S_1$ and $S_2$ and an infinite set of subcontinua $M_\infty$, $M_1$, $M_2$, $M_3$, $\ldots$ of $M$ satisfying the conditions of the Moore-Wilder lemma.\(^\dagger\) Let $S_3$ and $S_4$ be distinct hyperspheres concentric with $S_1$ and lying between $S_1$ and $S_2$. Each continuum $M_i$ contains a subcontinuum $K_i$ which contains a point $P_i$ on $S_3$ and a point $Q_i$ on $S_4$ and is a subset of the set $G$ consisting of $S_3$ and $S_4$ and all points which lie between $S_3$ and $S_4$. There exists a sequence of integers $n_1$, $n_2$, $\ldots$, such that $[P_{n_i}]$ has a sequential limit point $A$ and $[Q_{n_i}]$ has a sequential limit point $B$. By hypothesis there exists a finite set of continuous curves $C_1$, $C_2$, $C_3$, $\ldots$, $C_m$ which are subsets of $M$ and separate $A$ and $B$ in $M$.

**CASE I.** Suppose infinitely many of the continua $K_{n_i}$ contain a point of $\sum_{k=1}^{m} C_k$. As there are but a finite number of the curves $C_k$, one curve $C_k$ must contain a point $p_{n_i}$ of $K_{n_i}$ for infinitely many values of $i$. The set $[p_{n_i}]$ has a limit point $P$, which must belong to $M_\infty$ and to $G$. Let $\epsilon$ be a positive number such that no point of $S_1 + S_2$ is within a distance $\epsilon$ of $P$. As $C_k$, is a continuous curve, the point $P$

\(^*\) R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), pp. 254-260. Moore’s theorem is stated for two dimensions, but the extension to $n$ dimensions is obvious.

belongs to $C_k$ and there is a number $\delta_e > 0$ such that any point of $C_k$ whose distance from $P$ is less than $\delta_e$ can be joined to $P$ by an arc of $C_k$ of diameter less than $\epsilon$. There is a point $p_n$ of $[p_n]$ whose distance from $P$ is less than $\delta_e$. Let $\alpha$ denote an arc of $C_k$ with end points $P$ and $p_n$ and of diameter less than $\epsilon$. The arc $\alpha$ contains a point of $M_n$, and a point of $M_\infty$ and lies entirely between $S_1$ and $S_2$. By the Moore-Wilder lemma, $M_n$, is a component of the common part of $M$ and the set composed of $S_1$ and $S_2$ and all points lying between $S_1$ and $S_2$. Hence $M_n$, contains the arc $\alpha$. But this contradicts the condition of the lemma that $M_n$, and $M_\infty$ have no common points.

Case II. Suppose only a finite number of the continua $K_n$, contain points of $\sum_{k=1}^{m} C_k$. The set $M - \sum_{k=1}^{m} C_k$ is the sum of two mutually separated sets $M_A$ and $M_B$ containing $A$ and $B$ respectively. Every set $K_{n_i}$ which contains no point of $\sum_{k=1}^{m} C_k$ lies wholly in $M_A$ or wholly in $M_B$. There is an integer $j$ such that for $i \geq j$, the continuum $K_{n_i}$ contains no point of $\sum_{k=1}^{m} C_k$. Both $A$ and $B$ are limit points of the set $\sum_{i=j}^{m} K_{n_i}$. Either infinitely many of the sets $K_{n_i}$ $(i \geq j)$ belong to $M_A$ or infinitely many belong to $M_B$. If the first holds then $B$ is a limit point of $M_A$; under the second possibility the point $A$ is a limit point of the set $M_B$. In either possibility we have a contradiction since $M_A$ and $M_B$ are mutually separated.

The assumption that $M$ is not a continuous curve leads to a contradiction with the assumed condition in either case. Therefore the condition is sufficient.

It is to be noticed that in the proof of the necessity of the condition in Theorem 2 we showed that the separating continuous curves were bounded. Hence we have the following corollary and theorem.

**Corollary.** If $A$ and $B$ are points of a continuous curve $M$ lying in euclidean space of $n$ dimensions, there exists a subset of $M$ which consists of a finite number of bounded continuous curves and which separates $A$ and $B$ in $M$. 
Theorem 3. If $K_1$ and $K_2$ are any two mutually exclusive and closed point sets, one of which is bounded, then $K_1$ lies wholly in a finite number of the complementary domains of $K_2$.

Proof. Suppose the contrary is true. Then there exists an infinite sequence $D_1, D_2, D_3, \ldots$ of distinct complementary domains of $K_2$ each of which contains at least one point of $K_1$. For each positive integer $i$, let $P_i$ denote a point of $K_1$ belonging to $D_i$. Let $H$ denote the set of points $P_1 + P_2 + P_3 + \cdots$. By hypothesis either $K_1$ or $K_2$ is bounded. If $K_1$ is bounded, then $H$ is bounded because $H$ is a subset of $K_1$; and if $K_2$ is bounded, then since $H$ contains at most one point in the unbounded complementary domain of $K_2$, it readily follows that $H$ is bounded. Hence, in any case, $H$ is bounded; and since it is infinite, it must have at least one limit point $P$. Since $K_1$ is closed and contains $H$, it must contain the point $P$; and since $K_1$ and $K_2$ are mutually exclusive, $P$ must belong to some complementary domain $D$ of $K_2$. Clearly this is impossible, since $P$ is a limit point of $H$, and not more than one point of $H$ can belong to $D$. Thus the supposition that Theorem 3 is not true leads to a contradiction.

Theorem 4. In order that a continuum $M$ in a euclidean space $E_n$ of $n$ dimensions should be a continuous curve it is necessary and sufficient that every two mutually exclusive, closed, and bounded subsets of $M$ should be separated in $M$ by the sum of a finite number of subcontinua of $M$.

Proof.* The condition is sufficient. For suppose a continuum $M$ satisfies the condition but is not a continuous curve. Then by the Moore-Wilder lemma† it follows that there exist two different concentric hyperspheres $C_1$ and $C_2$ and a countable infinity of mutually exclusive subcontinua of $M$: $W, M_1, M_2, M_3, \ldots$ such that (1) if $D$ denotes the

* Compare this proof with that given by R. L. Moore for Theorem 1 of his paper, A characterisation of a continuous curve, loc. cit.
† See reference to the Moore-Wilder lemma above.
n-dimensional domain whose boundary is $C_1 + C_2$, then each of these continua contains at least one point on each of the hyperspheres $C_1$ and $C_2$, and each of them, save possibly $W$, is a component of the set of points $M \cdot (D + C_1 + C_2)$, and

(2) $W$ is the sequential limiting set of the sequence of continua $M_1, M_2, M_3, \cdots$. Let $A$ and $B$ denote the sets of points $W \cdot C_1$ and $W \cdot C_2$ respectively and, for each positive integer $i$, let $a_i$ denote the set of points $M_i \cdot C_1$ and $b_i$ the set $M_i \cdot C_2$. Since $A$ and $B$ are mutually exclusive, closed, and bounded subsets of $M$, by hypothesis there exists a subset $L$ of $M$ such that (1) $M - L$ is the sum of two mutually separated point sets $M_a$ and $M_b$ containing $A$ and $B$ respectively, and (2) $L$ is the sum of a finite number of continua $L_1, L_2, L_3, \cdots, L_m$. Since neither $A$ nor $B$ has a point in common with $L$, and $A$ contains no point of $M_b$ and $B$ contains no point of $M_a$, therefore there exist open sets $C_a$ and $C_b$, containing $A$ and $B$ respectively, such that $C_a$ contains no point of $L + M_b$ and $C_b$ contains no point of $L + M_a$. There exists an integer $\delta$ such that, for every $j$ greater than $\delta$, the point set $a_j$ lies wholly in $C_a$ and the point set $b_j$ lies wholly in $C_b$. Thus, for every $j$ greater than $\delta$, $M_j$ contains a point of $M_a$ and also a point of $M_b$. But $M_j$ is a subcontinuum of $M$, and every subcontinuum of $M$ which contains a point of each of the sets $M_a$ and $M_b$ must contain at least one point of $L$. Hence, for every $j$ greater than $\delta$, $M_j$ contains a point of $L$, and therefore of some one of the sets $L_1, L_2, \cdots, L_m$. It follows that there exists an integer $g$ and an infinite sequence of distinct positive integers $t_1, t_2, t_3, \cdots$ such that, for every $j$, $L_g$ contains at least one point in common with $M_{t_j}$. Since, for every $j$, the subcontinuum $L_g$ of $M$ contains a point of $M_{t_j}$ and a point of $M_{t_{j+1}}$ it follows by a lemma of R. L. Moore's* that $L_g$ must contain a point either of $a_{t_j}$ or of $b_{t_j}$. Thus there exists an infinite sequence of distinct integers $j_1, j_2, j_3, \cdots$, such that either $L_g$ has a point in common with each point set

* A characterization of a continuous curve, loc. cit., Lemma 2.
of the sequence \( a_1, a_2, a_3, \ldots \), or it has at least one point in common with each point set of the sequence \( b_1, b_2, b_3, \ldots \). In the first case it readily follows that \( A \) contains at least one point of \( L \), and in the second case that \( B \) contains at least one point of \( L \). But \( A + B \) is a subset of \( M - L \). Thus the supposition that \( M \) is not a continuous curve leads to a contradiction.

The condition is also necessary. For let \( M \) be any continuous curve in \( \mathbb{R}^n \), and let \( K_1 \) and \( K_2 \) be any two mutually exclusive, closed, and bounded subsets of \( M \). It follows by Theorem 3 that there exists a finite number \( D_1, D_2, D_3, \ldots, D_m \) of the complementary domains of \( K_2 \) whose sum contains the point set \( K_1 \). For each positive integer \( i \leq m \), let \( B_i \) denote the boundary of \( D_i \), let \( H_i \) be the set of points common to \( K_1 \) and \( D_i \), and let \( 4d_i \) be the minimum distance between the closed sets of points \( H_i \) and \( B_i \). For each point \( P \) of \( H_i + B_i \), let \( C_P \) denote a hypersphere with \( P \) as center and radius \( d_i \), and let \( G_i \) be the collection of all the hyperspheres \( [C_P] \) for all points \( P \) of \( H_i + B_i \). Since \( K_1 + K_2 \), and hence also \( H_i + B_i \), is bounded, then by the Borel theorem there exists a finite subcollection \( G_i \) of the hyperspheres of \( G_i \) such that \( H_i + B_i \) is a subset of the sum \( I_i \) of the interiors of the collection \( G_i \). Let \( T_i \) denote the point set \( (D_i + B_i) - I_i \). Then \( T_i \) is closed. Let \( F_i \) denote the sum of all the hyperspheres (not including their interiors) of the collection \( G_i \) which enclose at least one point of \( H_i \), and let \( N_i \) be the sum of all those which enclose at least one point of \( B_i \). Since the least distance between \( H_i \) and \( B_i \) is \( 4d_i \), and since the radius of each hypersphere of \( G_i \) is \( d_i \), it follows that \( F_i \) and \( N_i \) are mutually exclusive closed sets whose least distance apart is \( >d_i \). Let \( Q_i \) denote the collection of all those maximal connected subsets of \( M \) which lie wholly in \( T_i \) and contain at least one point of each of the sets \( F_i \) and \( N_i \). Each element of \( Q_i \) is a continuum, and since \( M \) is a continuous curve, it follows by the Moore-Wilder lemma that \( Q_i \) has just a finite number of elements. Hence \( Q_i \) is a finite
collection of mutually exclusive continua \( L_{1i}, L_{2i}, L_{3i}, \ldots, L_{ni} \) which belong to \( M \).

Now let \( L \) denote the point set \( \sum_{i=1}^{n} \sum_{j=1}^{n} L_{ji}. \) Then \( L \) is the sum of a finite number of mutually exclusive subcontinua of \( M \). Let \( M_a \) denote the sum of all those components of \( M - L \) which contain at least one point of \( A \), and let \( M_b \) denote the point set \( M - (M_a + L) \). No point of \( B \) belongs to \( M_a \). For if a point \( X \) of \( B \) belonged to \( M_a \), then* \( X \) could be joined in \( M \) to some point \( Y \) of \( A \) by an arc which contains no point of \( L \), and this arc would contain a subarc \( t \) which is a subset of some set \( T_i \) and which has its end points on \( F_i \) and \( N_i \) respectively; and the arc \( t \) would necessarily be a subset of some continuum of the collection \( Q_i \), contrary to the fact that \( t \) contains no point of \( L \). Therefore \( B \) must be a subset of \( M_b \). Since \( M \) is connected in kleinen and \( L \) is closed, it readily follows that \( M_a \) and \( M_b \) are mutually separated. Hence \( M - L \) is the sum of two mutually separated sets \( M_a \) and \( M_b \) containing \( A \) and \( B \) respectively, and therefore \( L \) separates \( A \) and \( B \) in \( M \).

**Theorem 5.** In order that a continuum \( M \) in a space of \( n \) dimensions should be a Menger regular curve† it is necessary and sufficient that every two points of \( M \) should be separated in \( M \) by some finite subset of \( M \).

**Proof.** The condition is sufficient. Let \( P \) be any point of \( M \) and \( \epsilon \) any positive number. Let \( C_1 \) and \( C_2 \) be two distinct hyperspheres each of which has \( P \) as center and is of radius less than \( \epsilon/4 \). Let \( D \) denote the domain between \( C_1 \) and \( C_2 \), and let \( K \) denote the set of points common to \( D + C_1 + C_2 \) and to \( M \). Then \( K \) is closed. Now by Theorem 2 it follows

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* R. L. Moore, *Concerning continuous curves in the plane*, loc. cit.
† A continuum \( M \) is said to be a Menger regular curve provided that for each point \( P \) of \( M \) and each positive number \( \epsilon \) there exists an open subset \( T \) of \( M \) of diameter less than \( \epsilon \) which contains \( P \) and whose \( M \)-boundary is finite. The \( M \)-boundary of an open subset \( T \) of a continuum \( M \) is the set of all those points of \( M - T \) that are limit points of \( T \). See K. Menger, *Grundzüge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1925–1926), pp. 276–306.
that $M$ is a continuous curve. By hypothesis, for each point $X$ of $K$ there exists a finite subset $N_x$ of $M$ which separates $X$ and $P$ in $M$. For each such point $X$, the maximal connected subset $H_x$ of $M - N_x$ which contains $X$ is an open subset of $M$ which does not contain $P$ and whose $M$-boundary is finite (a subset of $N_x$). Let $G_0$ denote the collection of sets $[H_x]$ for all points $X$ of $K$. Since $K$ is closed and bounded, then by the Borel theorem the collection $G_0$ contains a finite subcollection $G$ which covers $K$. Let $R$ denote the sum of all the point sets of the collection $G$. Then $K$ is a subset of $R$, and $R$ is an open subset of $M$. Furthermore $B$, the $M$-boundary of $R$, is finite, for $R$ is the sum of a finite number of the sets $H_x$. Now, supposing that $C_1$ is within $C_2$, let $A$ denote the set of all those points of $B$ which lie on or within $C_1$. Now $R + A$ does not contain $P$, for $P$ belongs to no set $H_x$ and to no $N_x$. Let $T$ denote the maximal connected subset of $M - A$ which contains $P$. It is readily seen that $T$ must lie within $C_1$. Hence the diameter of $T$ is less than $\epsilon$. The $M$-boundary of $T$ is finite, because it is a subset of $A$. Then, since $T$ is an open subset of $M$, it follows that $P$ is a regular point of $M$ and that $M$ is a Menger regular curve.

That the condition is necessary follows at once from the definition of a Menger regular curve.

**Theorem 6.** If every two points of a continuum $M$ are separated in $M$ by some finite subset of $M$, then every two mutually exclusive, closed, and bounded subsets of $M$ are separated in $M$ by some finite subset of $M$.

**Proof.** It follows by Theorem 5 that $M$ is a Menger regular curve. Then by a theorem of Menger's,* it follows that every two mutually exclusive, closed, and bounded subsets of $M$ can be separated in $M$ by some finite subset of $M$.

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* Loc. cit., Theorem 12.