THE POLAR CURVES OF PLANЕ ALGEBRAIC CURVES IN THE GALOIS FIELDS*

BY A. D. CAMPBELL

By imitating the proofs in Fine's College Algebra (pp. 460-462) and Veblen and Young's Projective Geometry (vol. I, pp. 255-256) we can readily show that also in the Galois fields of order \( p^n \) (\( p \) a prime integer) we have Taylor's expansion

\[
\begin{align*}
    f(x + \lambda X, y + \lambda Y, z + \lambda Z) &= f(x, y, z) + \frac{\lambda}{1!} (f_x' X + f_y' Y + f_z' Z) + \\
    &\quad + \frac{\lambda^2}{2!} (f_x'' X + f_y'' Y + f_z'' Z)^{(2)} + \cdots \\
    &\quad + \frac{\lambda^r}{r!} (f_x^r X + f_y^r Y + f_z^r Z)^{(r)} + \cdots + f(X, Y, Z) = 0,
\end{align*}
\]

where \((f_x^r X + f_y^r Y + f_z^r Z)^{(r)}\) is symbolic for an expression containing derivatives of the \( r \)th order, and \( f(x, y, z) = 0 \) is an algebraic curve of order \( n \). In the above expansion we must take all the derivatives as though \( p \) were not a modulus, cancel out common factors from numerators and denominators, and then set \( p = 0 \).

The \( r \)th polar of \((X, Y, Z)\) with respect to \( f(x, y, z) = 0 \) is

\[
\frac{1}{r!} (f_x^r X + f_y^r Y + f_z^r Z)^{(r)} = 0.
\]

In particular the \( r \)th polar of \((1, 0, 0)\) is \((1/r!)(\partial f(x, y, z)/\partial x^r) = 0\). We suppose first of all that \( n \) has the value

\[
n = \alpha p^m + \beta p^{m-1} + \cdots + \gamma p^2 + \delta p + \epsilon,
\]

where \( \epsilon \neq 0 \), \( p = \epsilon + \xi \), \( \xi \neq 0 \).

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We can write the polars of \((1, 0, 0)\) by a sort of detached coefficients, underlining the coefficients that have \(p\) as a factor, as follows:

\[
(1/1!)[n, n - 1, n - 2, \cdots, n - \varepsilon, n - \varepsilon - 1,
\quad \cdots, n - p - \varepsilon, \cdots, n - p^2 - \varepsilon, \cdots, 3, 2, 1] = 0,
\]

\[
(1/2!)[(n - 1), (n - 1)(n - 2), \cdots, (n - \varepsilon + 1)(n - \varepsilon),
\quad (n - \varepsilon)(n - \varepsilon - 1), \cdots, (n - p - \varepsilon + 1)(n - p - \varepsilon),
\quad (n - p - \varepsilon)(n - p - \varepsilon - 1), \cdots, (n - p^2 - \varepsilon - 1)
\quad \cdots, 3, 2, 1] = 0,
\]

\[
[1/(\varepsilon + 1)!][n - 1)(n - 2) \cdots (n - \varepsilon), (n - 1)(n - 2) \cdots
\quad (n - \varepsilon)(n - \varepsilon - 1), \cdots, (n - 2\varepsilon)(n - 2\varepsilon - 1), \cdots,
\quad (\varepsilon + 1)!] = 0,
\]

\[
(1/p!)[(n - 1) \cdots (n - \varepsilon) \cdots (n - p + 1) \cdots, p!] = 0,
\]

where \((n - \lambda) (n - \lambda - 1) \cdots (n - \lambda - i)\) stands for all the terms of the same \((n - \lambda - i - 1)\) power, which then have this common factor in their coefficients. From the above polars we see that the \(\varepsilon\)th polar has at \((1, 0, 0)\) a tangent having \((\varepsilon + 1)\)-point contact if \((1, 0, 0)\) is not on \(f(x, y, z) = 0\), otherwise a multiple point of order \(\varepsilon + 1\). The \((\varepsilon + 1)\)th polar, \((\varepsilon + 2)\)th, \(\cdots\), \((p - 1)\)th polar all have multiple points of order \(\varepsilon + 1\) at \((1, 0, 0)\). Similarly the \((p + \varepsilon + 1)\)th polar, \((p + \varepsilon + 2)\)th, \(\cdots\), \((2p - 1)\)th have at \((1, 0, 0)\) multiple points of order \(\varepsilon + 1\); also the \((2p + \varepsilon + 1)\)th polar points of order \((3p - 1)\) \(h\), \(\cdots\), the \((\theta p^i + \cdots + \phi p + \varepsilon + 1)\)th polar points of order \((\theta p^i + \cdots + \phi p + p - 1)\)th, etc. Moreover we note that if any one of the polar curves that have multiple points at \((1, 0, 0)\) is a curve of degree \(\varepsilon + 1\), then this polar curve is degenerate. Thus for \(p = 2\), \(n = 2^\varepsilon + 1\), \(\varepsilon = 1\), we find the 2d
polar is degenerate; for \( p = 3, n = 3 + 1, \epsilon = 1 \), we find again the 2d polar is degenerate.

If \( n = ap^m + \beta p^{m-1} + \cdots + \gamma p^2 + \delta p \), i.e. \( \epsilon = 0 \) in \( n \), then all the polars of \((1, 0, 0)\) pass through \((1, 0, 0)\) whether or not this point lies on \( f(x, y, z) = 0 \).

If \( n < p \) we find no peculiarities like the above.

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THE CHARACTERISTIC EQUATION OF A MATRIX*

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1. *Introduction.* Consider any square matrix \( A \), real or complex, of order \( n \). If \( I \) is the unit matrix, \( A - \lambda I \) is called the characteristic matrix of \( A \); the determinant of the characteristic matrix is called the characteristic determinant of \( A \); the equation obtained by equating this determinant to zero is called the characteristic equation of \( A \); and the roots of this equation are called the characteristic roots of \( A \). If \( A \) happens to be a matrix of a particular type certain definite statements may be made as to the nature of its characteristic roots. For example, if \( A \) is Hermitian its characteristic roots are all real; if \( A \) is real and skew-symmetric, its characteristic roots are all pure imaginary or zero; if \( A \) is a real orthogonal matrix, its characteristic roots are of modulus unity. However, if \( A \) is not a matrix of some special type, no general statement can be made as to the nature of its characteristic roots. In 1900 Bendixson† proved that if \( \alpha + i\beta \) is a characteristic root of a real matrix \( A \), and if \( \rho_1 \geq \rho_2 \geq \cdots \geq \rho_n \) are the characteristic roots (all real) of the symmetric matrix \( \frac{1}{2}(A + A') \), then \( \rho_1 \geq \alpha \geq \rho_n \). The extension to the case where the elements of \( A \) are com-

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