FACTORIAL SERIES IN TWO VARIABLES*

BY C. R. ADAMS

Because of its connection with the study of partial difference equations, the author has had occasion to examine the factorial series

$$\text{(A)} \qquad \sum_{n=0}^{\infty} \frac{(n!)^2 a_n}{x(x+1)\cdots(x+n)y(y+1)\cdots(y+n)}.$$

It may be of some interest to observe that the methods employed by Landau in his fundamental paper on factorial series† can immediately be extended to deduce the essential convergence properties of the series (A). We give here eight theorems corresponding to Theorems I–VIII of Landau. They may all be established by proofs parallel to his in every detail; some of them may also be inferred otherwise as indicated below.

THEOREM 1. *If the series* (A) *converges for the place*‡ (x_0, y_0), *it converges for any place* (x_1, y_1) *satisfying the conditions*§ $R(x_1) \geqq R(x_0)$, $R(y_1) \geqq R(y_0)$ (*both equality signs not to hold simultaneously*).

Only the following three possibilities may therefore occur: (a) the series (A) converges everywhere; (b) it converges nowhere; (c) there exist two associated real numbers λ_1, λ_2, such that the series converges for $R(x) \geqq \lambda_1$, $R(y) \geqq \lambda_2$ and diverges for $R(x) \leqq \lambda_1$, $R(y) \leqq \lambda_2$ (both equality signs not to hold simultaneously in either case). One would scarcely

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*Presented to the Society, April 6, 1928.

† *Über die Grundlagen der Theorie der Fakultätenreihen*, Sitzungsberichte der Münchener Akademie (math.-phys.), vol. 36 (1906), pp. 151-218.

‡We think of the complex variables x and y as represented by points in two distinct planes. Following a common terminology a pair of values x, y will be spoken of as the *place* (x, y).

§$R(x)$ is used to denote the *real part* of x.

expect the pair of numbers λ_1, λ_2 to be uniquely determined; the relation between them will presently be evident.

THEOREM 2. *The series* (A) *converges uniformly in the neighborhood of every place* (x, y) *satisfying the conditions* $R(x) > \lambda_1$; $R(y) > \lambda_2$; *and* $x, y \neq 0, -1, -2, \cdots$.

Then from a well known theorem in the theory of functions* follows

THEOREM 3. *The series* (A) *represents an analytic function within its related half-planes of convergence* $(x, y \neq 0, -1, -2, \cdots)$ *and in this region may be differentiated partially with respect to either variable as many times as may be desired.*

THEOREM 4. *The region of absolute convergence of the series* (A), *unless it converges everywhere or nowhere, is a pair of related half-planes bounded on the left by lines* $R(x) = \mu_1$, $R(y) = \mu_2$; *the lines themselves may be included or not.*

THEOREM 5. *If the series* (A) *converges for the place* (x_0, y_0) *and if we have* $R(x_1) > R(x_0) + a$, $R(y_1) > R(y_0) + b$, *and* $a + b = 1$, *the series* (A) *is absolutely convergent for the place* (x_1, y_1).

THEOREM 6. *The regions of convergence of the series* (A) *and of the Dirichlet series*

$$\text{(B)} \qquad \sum_{n=1}^{\infty} \frac{a_n}{n^{x+y}}$$

are the same; that is, for any place (x, y) $(x, y \neq 0, -1, -2, \cdots)$ *both series converge or both diverge.*

From this theorem, which is proved independently of the preceding ones, Theorem 1 could be inferred. In conjunction with Landau's Theorem VI it also forms the basis for the rather surprising result that *the regions of convergence of* (A) *and of the series*

$$\sum_{n=0}^{\infty} \frac{n! a_n}{(x + y)(x + y + 1) \cdots (x + y + n)}$$

*See Osgood, *Lehrbuch der Funktionentheorie*, vol. II, 1924, p. 13.

are identical. Hence we conclude that *the related abscissas of convergence of the series* (A) *have a constant sum.*

THEOREM 7. *The places of absolute convergence of the series* (A) *and* (B) *are the same.*

From Theorems 6 and 7 and Landau's Theorems VI–VIII follows at once

THEOREM 8. *If the sum of the abscissas of* [*absolute*] *convergence* $\lambda_1+\lambda_2[\mu_1+\mu_2]$ *is* $\geqq 0$, *it is given by*

$$\limsup_{t\to\infty}\frac{\log\left|\sum_{n=1}^{t}a_n\right|}{\log t}\quad\left[\limsup_{t\to\infty}\frac{\log\sum_{n=1}^{t}|a_n|}{\log t}\right].$$

It may be remarked that Landau's work can similarly be extended to prove the convergence properties of the series

$$\sum_{n=0}^{\infty}(n!)^m a_n/[x_1(x_1+1)\cdots(x_1+n)x_2(x_2+1)\cdots(x_2+n)\cdots x_m(x_m+1)\cdots(x_m+n)],$$

where m is any positive integer.

BROWN UNIVERSITY