# EVANS ON LOGARITHMIC POTENTIAL

*The Logarithmic Potential, Discontinuous Dirichlet and Neumann Problems.* By Griffith Conrad Evans. Colloquium Publications, American Mathematical Society, volume 6, 1927. viii+150 pp.

The appearance of this volume marks a broadening of the scope of the Colloquium Publications, in that these will no longer be confined to lectures given from time to time, by invitation of the American Mathematical Society, at its summer meetings. The new departure, however, could hardly have been initiated more smoothly or appropriately. For in the first place, Professor Evans has in the past been one of the Society's Colloquium lecturers, and secondly, his book is very closely related in character to the traditional Colloquium publications in that it is preeminently concerned with recent progress.

The book is characterized by the role played by Stieltjes and Lebesgue integrals and by functions of limited variation, and by the degree to which these concepts have enabled the author to express the conditions for his theorems in necessary and sufficient form. That these tools are particularly adapted to physical problems is a fact which the author early recognized. The material centers about a study of Poisson's integral in two dimensions and of the corresponding Stieltjes integral. The results are then extended to the integrals in terms of Green's functions for general regions.

In the introductory chapter, the fundamental properties of functions of limited variation and of Stieltjes integrals are succinctly developed, and some properties of Lebesgue integrals and theorems on the termwise integration of sequences are stated, with references or proofs. The last eight pages are given to Fourier series.

The second chapter develops properties of the Poisson-Stieltjes integral

$$\text{(A)} \qquad u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\phi - \theta)} dF(\phi),$$

where $F(\phi)$ is of limited variation, and of the Poisson integral

$$\text{(B)} \qquad u(r, \theta) = \frac{1}{2\pi}\int_0^{2\theta} \frac{1-r^2}{1 + r^2 - 2r\cos(\phi - \theta)} f(\phi)d\phi,$$

where $f(\phi)$ is summable.

The function $u(r,\theta)$ defined by (A) is harmonic ($r<1$); the function

$$F(r, \theta) = \int_0^{\theta} u(r, \theta)d\theta$$

approaches, as $r \longrightarrow 1$, $\theta$ being fixed, the value

$$\tfrac{1}{2}\{F(\theta + 0) + F(\theta - 0)\} - \tfrac{1}{2}\{F(0 +) + F(0 -)\};$$

$u(r,\theta)$ is the difference of two non-negative functions for $r<1$; $u(r,\theta)$ is a function of $\theta$ of limited variation, uniformly as to $r$; at every point $P(1,\phi)$ at which $F(\phi)$ has a derivative $F'(\phi) = f$ (i.e., almost everywhere), $u(r,\theta) \rightarrow f$

as $r\to 1$, under the sole restriction that $(r,\theta)$ remain between two chords of the unit circle terminating in $P$. These are among the properties derived. They are paralleled by those of the function $u(r,\theta)$ defined by (B).

The third chapter centers in the fundamental theorem, stated as follows in terms of the above function $F(r,\theta)$, and an infinite sequence

$$r_1, r_2, r_3, \cdots, r_n < 1, \lim r_n = 1.$$

(1) If $u(r,\theta)$ is harmonic for $r<1$, and if $F(r_n,\theta)$ is of limited variation as a function of $\theta$, uniformly as to $n$, then $u(r,\theta)$ admits the representation (A).

(2) If $u(r,\theta)$ is harmonic for $r<1$, and if $F(r_n,\theta)$ is absolutely continuous as a function of $\theta$, uniformly as to $n$, then $u(r,\theta)$ admits the representation (B).

This theorem, in combination with Chapter 2, gives necessary and sufficient conditions that a harmonic function admit the representation (A) or (B), as well as a considerable number of properties of such harmonic functions. A section is devoted to interesting special cases of functions given by (B). Of course the results of Chapter 2 include existence proofs of a very general character for the Dirichlet problem for the circle, and Chapter 3 then furnishes the desirable complements in the way of proofs of uniqueness. The formulation of these results constitutes the closing section of Chapter 3.

Chapter 4 is concerned with the potential of a simple distribution on the circumference of a circle of radius $a$:

$$\text{(C)} \qquad v(r, \theta) = -\frac{a}{2\pi}\int_0^{2\pi} \log\,[a^2 + r^2 - 2ar\cos(\phi - \theta)]dF(\phi) + A,$$

where $F(\phi)$ is periodic and of limited variation. A necessary and sufficient condition that $v(r,\theta)$, harmonic for $r<a$, admit a representation (C) is that

$$\int_0^{2\pi} \left| \frac{\partial}{\partial r} v(r, \theta) \right| d\theta$$

be bounded. Other properties of $v(r,\theta)$ follow, among them that

$$\lim_{r\to a}\int_0^{\theta} \frac{\partial v}{\partial r}\, d\theta = F(\theta) - F(0).$$

But this is in effect the statement that the flux of the vector field whose potential is $v$, across any given portion of the boundary, is a given quantity, and so we are led to a solution of the problem of Neumann having validity even when the normal derivative of $v$ does not exist on the boundary. The chapter is concluded by a study of the possibility of representing a function of the complex variable, analytic within a circle, by the Cauchy integral formula, the integral being extended over the circumference.

Chapter 5 contains the extension of results for the circle to the general simply connected region of the plane with more than one boundary point, and Chapter 6 to regions of finite connectivity. The theorems are stated in a form invariant under conformal transformation in terms of Green's function and its conjugate, but careful reasoning is necessary to establish them. They are substantial generalizations, not merely obvious ones. Incidentally Chapter 6 contains some interesting results on isolated singularities of harmonic functions. The theorem ascribed to Lebesgue, by the way, on page 86, is due to Bôcher (see the reference on page 111).

The last chapter is a development of Professor Evans' symposium at the meeting of the Southwestern Section of the American Mathematical Society in November, 1926. At the beginning it reacts to simply connected regions, and establishes, among others, the result that the analytic transformation which carries the interior of the unit circle into the interior of a region with rectifiable boundary on a Riemann surface is conformal almost everywhere on the boundary. For regions of infinite connectivity a theorem is established which is concerned with the unique determination by its boundary values of a function harmonic and bounded in a region of infinite connectivity. The chapter closes with a discussion of convergence in the mean of various orders, relating a theorem of Noaillon with those of the author, and with a section on the characterization of harmonic functions by integral properties.

There are some thirty exercises at intervals throughout the text. These are designed in part to familiarize the reader with the concepts introduced, and in part to complement the theory, and they are to be taken seriously. This does not mean, however, that the book is to be regarded as a text, or a systematic introduction to potential theory. For satisfactory results, the reader should know something of potential theory, functions of a complex variable, and Lebesgue integrals. The author's hope, expressed in the preface, that the book will be found suitable for graduate students of a moderate degree of advancement will be fulfilled in the case of those with initiative: potential investigators. Others will have difficulty in supplying the details of reasoning suppressed by the author's succinct style. It is for investigators that the book will find its greatest usefulness, whether at the beginning of their careers or later.

In general, and in all essentials, the book is written with the care and thoroughness we should expect of its author. There are a few minor points where oversights have occurred. Thus Corollary 3 on page 7 is wrong, and the word uniformly has been omitted from the phrase "and (4) is convergent" in the last line of the second paragraph on page 126. These are the only cases noticed worth mentioning, and nothing is affected in the sequel.

The theory presented is Professor Evans' creation, most of the theorems developed being the results of his own researches. These here find a systematic and consecutive presentation. The arrangement is excellent. In spite of a well rounded character of the investigations set forth, the book leaves the reader with a sense of vistas opened. The methods and results cannot fail to influence further work on potential theory, Taylor's series, and elliptic differential equations. It will be indispensable to those occupied in these fields, and will hold an honored place in the colloquium publications.

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