CONCERNING LOGICAL PRINCIPLES*

BY C. H. LANGFORD

1. *Introduction.* In a recent number of this Bulletin, Dr. Alonzo Church discusses certain questions concerning the nature of the principles of logic, and advances certain views with regard to the nature and status of these principles, in particular, with regard to the status of the so-called principle of excluded middle.† It would seem that the views which Dr. Church is holding are views that are often held concerning the principles of logic, or similar to views that are often held; but it is clear that they are incompatible with certain tenets of ordinary logic that are commonly accepted; and I think it possible that those who adopt positions similar to the one Dr. Church appears to be adopting have not considered in detail the bearing of such positions on more ordinary logical conceptions, and have not assured themselves that the views they are holding are in fact compatible with other views which they would be equally inclined to accept. For this reason I wish to present, as clearly and as briefly as I can, some points concerning the nature and status of the principles of logic, and to suggest an account of these principles which is in accordance with commonly accepted tenets of ordinary logic, and which is incompatible in many respects with the interpretation of logical principles suggested by Dr. Church. In giving this account I shall be concerned on occasion to point out explicitly the bearing of the views I shall be advocating on views advanced by Dr. Church, but for the most part I shall be confined simply to presenting a different interpretation.

2. *Alternative Logics.* We may begin by considering the way in which logical principles are exemplified in the relation-

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† *On the law of excluded middle*, this Bulletin, vol. 34 (1928), pp. 75-78.

ships in which propositions stand, one to another. We may consider, in particular, the way in which such principles are exemplified in a pair of propositions having respectively the forms $(x).fx$ and $(\exists x).\sim fx$, and we may use these expressions to denote the propositions in question. These propositions exemplify the principle of excluded middle, in that $(x).fx.\vee.(\exists x).\sim fx$ could not be false; they exemplify the principle of contradiction, in that $(x).fx.(\exists x).\sim fx$ could not be true; they exemplify logical necessitation or entailing, in that $\sim(x).fx$ entails $(\exists x).\sim fx$; they exemplify logical equivalence, in that $\sim(x).fx.\equiv.(\exists x).\sim fx$ could not be false; and, no doubt, they exemplify other logical principles and relations, which are perhaps nameless, but which we might discover and name. Now one point which is brought out by an examination of this example, and upon which I wish to place emphasis, is that the logical principles and logical relations which these propositions exemplify in the relationship in which they stand, are found there as a matter of discovery; so that the occurrence of these properties is not at all an occasion for the exercise of choice or preference on our part. Another point of equal importance, connected with the first, is that these logical properties are what may be called necessary properties, in that their occurrence is dependent upon characteristics essential to the being of the propositions themselves, and upon nothing else; but in order to bring out this point, it will be necessary to describe a distinction which is commonly known as the distinction between internal and external relations.

When we consider the relations which hold among entities of various sorts, there appears to be a fundamental division among these relations, which we can describe by saying that the occurrences of some relations are dependent solely upon intrinsic features of the terms related, whereas other relationships are fortuitous so far as the intrinsic features of the terms related are concerned; or by saying, as is often done, that some relationships are grounded in the nature of their terms and others are not. Thus, to use a simple illustration, this pen, with which I am writing, is related to the paper upon which I am writing in a way that we can describe by saying that the pen is in contact with the paper. But it is clear that there is nothing in our conception of the terms having this relation from which the fact that they are so related could be inferred; and we can express this circumstance with regard to the relationship in which these things stand by saying that their being so related is conceptually fortuitous, or by saying that they stand in external relationship in this respect. On the other hand, there are two facts connected with this pen which are related in a way that is not conceptually fortuitous, namely, the fact that this pen is green, and the fact that it is colored. These facts are related in such a way that the first necessitates the second; and this is an internal relationship.*

Now we are often interested in making suppositions that are contrary to fact; and such suppositions are sometimes possible and sometimes not, and whether they are possible depends upon the nature of the facts in question. Thus we can make a supposition contrary to fact by supposing that this pen is not in contact with this paper, and this supposition is intelligible, and might be of interest; but when, for example, we attempt to suppose that this pen is both green and not colored, we find that our assent to this attempted supposition is merely verbal, that our words cannot retain their meanings, since being green involves being colored; and we are implicated in a species of self-contradiction, and consequent unintelligibility. Of course we can *say* that this pen is both green and not colored, and understand that what we say could not be true, that is, that there is no intelligible supposition to the effect that it is true. Now wherever internal relationships occur, there facts occur which do not admit of suppositions contrary to them; and such facts are known as analytic facts. On the other hand, wherever ex-

*The best discussion of internal and external relations with which I am acquainted is that of Professor G. E. Moore, in his *Philosophical Studies*, Chap. 9.

ternal relationships occur, there facts occur which do admit of suppositions contrary to them; and such facts are known as contingent or empirical facts.

This brings me to a point that I want to make concerning the possibility of alternatives to ordinary logical principles that shall be in some sense incompatible with these principles. It is clear that all of the facts pointed out above, with regard to the relationship in which the two propositions $(x).fx$ and $(\exists x).\sim fx$ stand, are analytic facts; and generally, it is clear, I think, that all logical facts are analytic, and thus that logical principles, which are based on logical facts, do not admit of intelligible alternatives. This means that we cannot have alternative logics; for logic is the system of all propositions expressing analytic facts of a certain kind, namely formal analytic facts, and there cannot in the nature of the case be more than one such system, actual or conceivable.

3. *Deduction*. We may now examine in some detail the nature of the connections existing between propositions, and between properties, in virtue of which we are able to argue validly from one proposition to another, or from one property to another. When we consider a pair of properties, p, q, it often happens that they are related in a way which we can describe by saying that there could not be an instance of the first that is not also an instance of the second; and when this is the case, we say that the first entails or necessitates the second, that the second is deducible from the first.* As an example of this relation, we may take a case of entailing that occurs in connection with certain properties which are similar to properties commonly used in the definition of serial order, but which differ from properties of serial order in that reference to a class K, that is, to a function fx, is omitted. It is clear that the conjunction of the property of transitivity with the contrary of the reflexive

*See Lewis, *A Survey of Symbolic Logic*, Chap. 5.

property necessitates the property "$\sim Rx_1x_2 \vee \sim Rx_2x_1$ for every x_1, x_2"; that is, that

$$(x_1, x_2, x_3): Rx_1x_2 . Rx_2x_3 . \supset . Rx_1x_3 :. (x) . \sim Rxx$$

entails*

$$(x_1, x_2) . \sim Rx_1x_2 \vee \sim Rx_2x_1 .$$

As soon as we have envisaged clearly the meanings of these two properties, we see that the first involves the second, and that this relation is immediate and direct, that no mediating principles are required. This case is relatively simple, but in more complicated cases the situation is precisely the same; in complicated cases we often require an elaborate technique of proof, for the purpose of exhibiting to ourselves the relatedness of the properties in question, but this technique of proof is employed solely for the purpose of displaying the facts, and does not in any way condition them. Moreover, the fact that the first of the above properties necessitates the second, so far from being dependent upon some logical principle, is itself of such a kind that it can be taken as the ground of a logical principle, which we can express by saying that there could not be an instance of the propositional function

$$(x_1, x_2, x_3): Rx_1x_2 . Rx_2x_3 . \supset . Rx_1x_3 :. (x) . \sim Rxx :. \supset . (x_1, x_2) . \sim Rx_1x_2 \vee \sim Rx_2x_1$$

that is false, just as we can express the principle of excluded middle by saying that there could not be an instance of the propositional function $p \vee \sim p$ that is false. This new principle differs from the principle of excluded middle in being more determinate; but that is a matter of degree, not an essential difference. In general, then, whatever properties $p_1, \cdots, p_n$ may be, if q is a logical consequence of $p_1 \cdots p_n$,

*It is to be noted that R is not an undefined term, but a logical function, $f(x, y)$; or perhaps it would be more accurate to say that R and similar entities, which are often called undefined terms, are simply logical functions or other items of logical structure. See Russell, *Introduction to Mathematical Philosophy*, Chap. 3.

we can say that q's being a consequence of $p_1 \cdots p_n$ does not depend upon anything outside the properties themselves, and especially that it does not depend upon anything that could be described as the logic which we assume, since the fact of q's following from $p_1 \cdots p_n$ is itself an analytic fact, namely, the fact that there could not be an instance of $p_1 \cdots p_n \cdot \sim q$ that is true.

I think it possible that the view that logical principles are in some sense premises in the deduction of one property from another, or of one proposition from another, is responsible for Dr. Church's view that we can have logics in which some ordinary logical principle is said to be not assumed, or in which some logical principle is invalid; for of course it often happens that a proposition $p_1 . p_2$ entails a proposition q, whereas p_1 alone does not entail q; and if p_2 were thought of as being some logical principle, then it might still be held that q could not be inferred if p_2 were not asserted, and that if p_1 were asserted and q denied, p_2 would have to be denied. In opposition to this view, I am maintaining that if we attempt to deny some logical principle, which is in fact valid, and on the basis of this deny certain inferences from one proposition to another, we do not get a new logic—we simply make a mistake; and similarly, that if we refuse to recognize an inference from one proposition to another, on the ground that we do not admit the logical principle which these propositions exemplify in the relationship in which they stand, then we simply make a mistake of another kind.

And I should like to introduce a remark at this point with regard to another view suggested by Dr. Church, a view to the effect that in logic we are not concerned with questions as to the truth and falsity of propositions. We do not raise questions concerning the truth and falsity of premises at the basis of abstract deductive systems, for the sufficient reason that, being properties, not propositions, these premises are neither true nor false, and we do not raise questions concerning the truth and falsity of propositions that are instances of properties at the basis of abstract deductive systems, because these propositions, when they are true, are not logical truths; but we do raise questions concerning the truth of propositions which we take to be logical principles, since these propositions must be true if we are not to be mistaken in supposing that they are logical principles.

In what has been said up to this point, and especially in what has been said concerning analytic and empirical facts, and internal and external relations, there is implicit a fundamental division of properties into two classes,—the class of contingent properties on the one hand, and the class of necessary and impossible properties on the other. An ordinary property, such as $f(x, y)$, is a contingent property if and only if, in view of its structure, it could have an instance that is true, and could have an instance that is false; whereas, a property is necessary if and only if it is such that it could not have an instance that is false, and a property is impossible if and only if it is such that it could not have an instance that is true. Thus, if we form the property $f(x, y) \vee \sim f(x, y)$, we have a necessary property, which is a specific form of the property involved in the principle of excluded middle, whereas if we form the property $f(x, y) . \sim f(x, y)$, we have an impossible property, which is a specific form of the property involved in the principle of contradiction. In this connection I wish to suggest, as a possible extension of the analysis given of logical and mathematical propositions of the form "p implies q," that any proposition of logic or mathematics can be expressed as an assertion with regard to some property that that property could not have an instance that is false (and, alternatively, as an assertion with regard to a property that that property could not have an instance that is true), or as an assertion with regard to some property that that property could have an instance that is true (and, alternatively, as an assertion with regard to a property that that property could have an instance that is false,—as in theorems on non-deducibility), or as a combination of such assertions. In sets of properties at the basis of deductive systems, and often elsewhere, contingent properties are given in isolation from the propositions into which they enter; but necessary properties do not as a rule occur in isolation. Thus, in an arithmetical proposition, say $2+3=5$, we have a necessary property whose instances express relationships of classes, and we have no conventional way of expressing this property as distinguished from the logically necessary proposition into which it enters; but if we allow $f(2, 3, 5)$ to express the arithmetical property in question, then we can, so I am holding, express the arithmetical proposition by saying that there could not be an instance of $f(2, 3, 5)$ that is false. As illustrations of contingent logical properties, we may take fx, Rxy, the cardinal number 10, the order-type ω, and the system of abstract euclidean geometry. Of course a property such as either of the last two is not in practice dealt with directly, but through a logically equivalent property, called a postulate-set or set of defining properties.

Now some properties are species of other properties, in the sense in which *green* is a species of *color*, or in which *roundness* is a species of *shape*; and this relation of species to genus is relevant to an account of the way in which propositions that are known as logical principles are related to other logical and mathematical propositions. I am holding that propositions known as logical principles do not differ in any essential respect from logical and mathematical propositions generally; but they are, as a rule, relatively simple propositions, so that their truth can be apprehended more immediately, and this is no doubt a reason for their being selected as principles. There is, however, another reason for their being selected, connected with their simplicity: the properties upon which logical principles are based, being simple, are more generic, and have many species among less generic properties, so that logical principles have many consequences among less generic propositions. Thus $(x)fx \vee (\exists x)\sim fx$ is a species of $p \vee \sim p$, and is sufficient for $p \vee \sim p$; but we do not formulate principles such as

"$(x)fx \vee (\exists x) \sim fx$ could not have an instance that is false" (or "$(\exists x) \sim fx \cdot (x)fx$ could not have an instance that is true"), because a single more generic principle will do.

It is to be noted, however, that whether we consider logical principles at all is ultimately a matter of preference, and is dependent upon whether we wish to organize logical and mathematical propositions into a deductive order—as is done, for example, in *Principia Mathematica*. One who is not interested in such organization can insist that any logical or mathematical proposition, say the proposition to the effect that π is transcendental, is just as much a logical principle as the principle of excluded middle, since the evidence for the truth of the proposition is found within the proposition itself. Of course this does not mean that relationships of deducibility which occur among logical propositions are in any sense arbitrary; it means that they are not logically prior to relationships of deducibility which occur among other propositions, and thus that it is quite unnecessary to infer the implication of one contingent proposition by another from the implication of one logical proposition by another. For when we consider the logical propositions to which the relationships of necessary properties give rise, we see that these propositions are species of ordinary logical principles, and are thus the same in kind as the logical propositions which arise from relationships of contingent properties. On the other hand, we have noted that properties which are logically impossible can be used equally well in the formulation of logical principles, and here too it is clear that the relations of an impossible property to another property—whether this other property is impossible, necessary, or contingent—simply give rise to species of ordinary principles.

We may now revert briefly to the point originally made concerning the impossibility of alternatives to logical principles, in order to connect that point with the subsequent discussion. It is clear that if the account given of logical principles is a right account, then, in view of the way in which these principles are formulated, what precisely distinguishes a logical principle from an ordinary contingent proposition is the absence of alternative possibilities, this impossibility of alternatives being explicitly stated in the formulation of the principle itself; so that whoever holds with regard to an assigned proposition that there could be circumstances under which that proposition would fail is holding that the proposition in question is not a logical principle.

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NOTE ON EINSTEIN'S EQUATION OF AN ORBIT

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In a paper* bearing the above title Morley has given an extremely elegant solution of Einstein's equation

$$(1) \qquad \left(\frac{dx}{d\theta}\right)^2 = 2x^3 - x^2 + 2\lambda x - \lambda^2(1 - e^2),$$

which defines the motion of a single planet about the sun. Here r, θ are the polar coordinates of the planet, a the major semi-axis, e the eccentricity of the orbit, M the mass of the sun, and

$$(2) \qquad x = \frac{M}{r}, \qquad \lambda = \frac{M}{a(1 - e^2)} \cdot$$

In Eddington units, $M = 1.45$. For Mercury, the values are

$$a = 5.8 \cdot 10^{-7}, \qquad e = 0.206, \qquad \lambda = 2.6 \cdot 10^{-8}.$$

The roots of the right side of (1) are thus, to a high degree of approximation,

$$(1 - e)\lambda, \qquad (1 + e)\lambda, \qquad \tfrac{1}{2} - 2\lambda.$$

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* American Journal of Mathematics, vol. 43 (1921), p. 29. I notice two obvious typographical errors in this paper. In the last term of (2) α should be α^2; also just below, x_1 should read $x_1 = \frac{1}{2} - 2\alpha$.