PLANE NETS WHOSE FIRST AND MINUS FIRST LAPLACIAN TRANSFORMS EACH DEGENERATE INTO A STRAIGHT LINE*

BY J. O. HASSLER

1. *Introduction.* In the projective theory of plane nets, as developed by Wilczynski,† the members $y_1(u, v)$, $y_2(u, v)$, $y_3(u, v)$ of any fundamental system of solutions of a completely integrable system of partial differential equations of the form

$$(1)\qquad \begin{cases} y_{uu} = ay_u + by_v + cy, \\ y_{uv} = a'y_u + b'y_v + c'y, \\ y_{vv} = a''y_u + b''y_v + c''y \end{cases}$$

are interpreted as the homogeneous coordinates of a point P_y in a plane, defining a non-degenerate net of plane curves consisting of two one-parameter families. The invariants and covariants of the system (1) under the transformations

$$(2)\qquad y = \lambda(u,v)\bar{y},$$

and

$$(3)\qquad \bar{u} = U(u), \text{ and } \bar{v} = V(v)$$

may be interpreted geometrically by certain projective properties of the net. The two covariants

$$(4)\qquad \rho = y_u - b'y, \quad \sigma = y_v - a'y$$

may be taken as the homogeneous coordinates of two points P_ρ and P_σ when we substitute successively for y the values y_1, y_2, and y_3. As u and v vary P_σ and P_ρ describe nets called the first and minus first Laplacian transforms of the original net.

---

* Presented to the Society, April 6, 1928.

† E. J. Wilczynski, *One-parameter families and nets of plane curves*, Transactions of this Society, vol. 12 (1911), pp. 473–510.

In this paper we consider nets such that the curves $u=\text{const.}$ of the first Laplacian transform all degenerate into one straight line while the curves $v=\text{const.}$ go into points on the line, and the curves $v=\text{const.}$ of the minus first Laplacian transformd egenerate into one straight line while the curves $u=\text{const.}$ go into points on the line. The invariants and covariants of the net are computed and the net determined by certain boundary conditions.

2. *The Canonical Form of the Equations and the Invariants.* If we apply the transformation (2) to the system of equations (1), we obtain

$$(1')\qquad \begin{cases} y_{uu} = -B'y_u + By_v + Cy, \\ y_{uv} = A'y_u + B'y_v + C'y, \\ y_{vv} = A''y_u - A'y_v + C''y, \end{cases}$$

where the coefficients B', B, C, etc., are functions of a, b, c, a', b', c', a'', b'', c'', λ and their derivatives.* The system is then said to be in its canonical form. The coefficients B and A'' are relative invariants of the system under the transformations (2) and (3). The five remaining fundamental invariants are

$$(5)\qquad \begin{aligned} &\mathfrak{A}' = A' + \frac{A_v''}{6A''}, \quad \mathfrak{B}' = B' + \frac{B_u}{6B}, \quad \mathfrak{C}' = C' + A'B', \\ &\mathfrak{C} = C - B_u' - 2B'^2 + A'B, \\ &\mathfrak{C}'' = C'' - A_v' - 2A'^2 + A''B'. \end{aligned}$$

Two other invariants,†

$$(6)\qquad H = C' + A'B' - A_u', \quad K = C' + A'B' - B_v',$$

have a special significance in connection with the Laplacian transforms.

Wilczynski has shown‡ that if $H=0$, the curves $v=\text{const.}$ of the first Laplacian transform degenerate into points and

* Wilczynski, loc. cit., equations (13).

† Wilczynski, loc. cit., equations (21) and (29).

‡ Wilczynski, loc. cit., pp. 489–490.

the curves $u=$const. all coincide with the locus of these points, making the degenerate transformed net consist of a single curve. Similarly, if $K=0$, the minus first Laplacian transform becomes a single curve $v=$const. Furthermore, if*

$$A''^2K + A''\mathfrak{C}_v'' - \mathfrak{C}''A_v'' = 0, \tag{7}$$

the curves $u=$const. of the first Laplacian transform become straight lines, and if

$$B^2H + B\mathfrak{C}_u - \mathfrak{C}B_u = 0, \tag{8}$$

the curves $v=$const. of the minus first Laplacian transform become straight lines. We shall consider the case where equations (7) and (8) are satisfied simultaneously with

$$H = K = 0. \tag{9}$$

From (7) and (8), by means of (9), we obtain, after integrating,

$$\log\frac{\mathfrak{C}''}{A''} = \phi(u), \qquad \log\frac{\mathfrak{C}}{B} = \psi(v),$$

where ϕ and ψ are arbitrary functions of u and v, respectively. The quotients $\mathfrak{C}''/A''$ and $\mathfrak{C}/B$ are relative invariants. If we apply the transformation (3), they are transformed into

$$\frac{\bar{\mathfrak{C}}''}{\bar{A}''} = \frac{1}{U'}\,\frac{\mathfrak{C}''}{A''}, \qquad \frac{\bar{\mathfrak{C}}}{\bar{B}} = \frac{1}{V'}\,\frac{\mathfrak{C}}{B}.$$

If we choose U and V so that $U'=e^{-\phi(u)}$ and $V'=e^{-\psi(v)}$, we have

$$\mathfrak{C}'' = A'', \qquad \mathfrak{C} = B. \tag{10}$$

The system (1) is subject to certain integrability conditions† which, if used with (5), (6), (9), and (10), enable us to express all of the coefficients of the canonical form in terms of B and A''. In fact, we find that

* Wilczynski, loc. cit., equation (54a).

† Wilczynski, loc. cit., equations (5) and (14).

$$
(11)\quad \begin{cases} B = B, \quad C = \dfrac{2}{9}(1 + B + B^2) - \dfrac{1}{3}(B_u + B_v), \\ A' = \dfrac{1-A''}{3}, \; B' = \dfrac{1-B}{3}, \; C' = \dfrac{1}{9}(2A''B + A'' + B - 1), \\ A'' = A'', \quad C'' = \dfrac{2}{9}(1 + A'' + A''^2) - \dfrac{1}{3}(A_u'' + A_v''), \end{cases}
$$

with the further conditions that the invariants A'' and B must satisfy the partial differential equations

$$
(12)\qquad A_u'' = B_v = -A''B.
$$

The general solution of (12) is*

$$
(13)\qquad A'' = \frac{\psi'(v)}{\phi(u) + \psi(v)}, \quad B = \frac{\phi'(u)}{\phi(u) + \psi(v)},
$$

where ϕ and ψ are arbitrary functions of u and v, respectively. It is easy to verify that the integrability conditions are identically satisfied if a system of partial differential equations of the form (1′) has coefficients defined by (11) and (13). Hence we may state the following conclusion.

If the coefficients of a system of partial differential equations of the form (1′) *satisfy conditions* (11) *and* (13), *any fundamental system of solutions* y_1, y_2, y_3 *of* (1′) *defines a net whose first and minus first Laplacian transforms each degenerate into a straight line. Conversely, any net whose first and minus first Laplacian transforms each degenerate into a straight line gives rise to a system of partial differential equations of the form* (1′) *whose coefficients satisfy conditions* (11) *and* (13).

3. *Determination of the Net by Boundary Conditions.* In the projective differential geometry of plane curves† three linearly independent solutions $y_1(x)$, $y_2(x)$, $y_3(x)$, of a linear homogeneous differential equation of the third order,

---

* Acknowledgement is hereby given to G. E. Raynor, of the University of Oklahoma, for the solution of these equations.

† E. J. Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, Leipzig, B. G. Teubner, 1916, pp. 58–61.

$$\frac{d^3y}{dx^3} + 3p_1\frac{d^2y}{dx^2} + 3p_2\frac{dy}{dx} + p_3y = 0, \tag{14}$$

are interpreted as the homogeneous coordinates of a point on a plane.

Two functions of the coefficients of (14), called θ_3 and θ_8 by Wilczynski, which remain invariant under the transformations

$$y = \lambda(x)\eta, \qquad x = f(\xi),$$

are sufficient to determine the projective properties of the curve. If, then, two arbitrary functions $\omega_3(x)$ and $\omega_8(x)$ be given there exists a curve whose invariants θ_3 and θ_8 are respectively equal to these given functions and this curve will be uniquely determined except for projective transformations. In particular, if $\theta_3 = 0$, the corresponding curve is a conic.

If we differentiate both members of the first equation of (1′) with respect to u and then eliminate y_v and y_{uv}, we obtain an equation of the form (14). We may then compute the invariants θ_3 and θ_8 for a curve $v = \text{const.}$ of the net under consideration. In fact, if we employ the equations derived by Wilczynski, making use of equations (11) and (13), we find

$$\begin{aligned} 54\theta_3 = & -9\frac{\phi^{\text{iv}}}{\phi'} + 9\frac{\phi'''}{\phi'} + 6\frac{\phi''}{\phi'} + 45\frac{\phi''\phi'''}{(\phi')^2} \\ & - 15\frac{\phi''^2}{\phi'^2} - 40\frac{\phi''^3}{\phi'^3} + 4, \\ \theta_8 = & \; 6\theta_3\theta_3'' - 7\theta_3'^2 - 27P_2\theta_3^2, \end{aligned} \tag{15}$$

where the accents indicate derivatives with respect to u and where

$$P_2 = p_2 - p_1^2 - p_1' = \frac{\phi'''}{3\phi'} - 4\frac{\phi''^2}{9\phi'^2} - \frac{\phi''}{9\phi'} - \frac{1}{9}\,.$$

We notice that the invariants are independent of v, since ϕ is a function of u alone. All the curves $v = \text{const.}$ are projectively equivalent.

Let C be any analytic curve, not a straight line, and let equation (14) be its differential equation with respect to any independent variable x to which it may be referred parametrically. Let $\theta_3=\omega_3(x)$ and $\theta_8=\omega_8(x)$ be the values of its invariants as functions of x. If the curve $v=v_0$ of the net coincides with C it must be possible to determine u as a function of x so that the equations

$$(16)\qquad \begin{cases} \theta_3(u) = \omega_3(x), \\ \theta_8(u) = \omega_8(x) \end{cases}$$

are identically satisfied. On the other hand, if equations (16) are identically satisfied the curve C will be projectively equivalent to the curve $v=v_0$ of the net.

To reduce the order of the differential equations we make the substitution

$$(17)\qquad w = \frac{\phi''}{\phi'}$$

in equations (15). If we substitute the values of θ_3 and θ_8 from (15) in (16) we find that the differential equations to be satisfied by w and u as functions of x have the form

$$(18)\qquad \begin{cases} -9\dfrac{d^2w}{du^2} + 9(1+2w)\dfrac{dw}{du} - 4w^3 - 6w^2 \\ \qquad\qquad\qquad +6w + 4 = 54\omega_3(x), \\ 6\omega_3\dfrac{d^2\omega_3}{du^2} - 7\left(\dfrac{d\omega_3}{du}\right)^2 - 3\left(\dfrac{dw}{du} - w^2 - w - 1\right)\omega_3^2 = \omega_8(x). \end{cases}$$

If C be not a conic, so that $\omega_3(x)\neq 0$, we can by a suitably chosen transformation of the independent variable x make $\omega_3=1$.* We may then, after using x as the independent variable in all the differentiation, write the equations in the form

* Wilczynski, *Projective Differential Geometry*, p. 61.

$$
(19)\quad \begin{cases} -9\dfrac{d}{dx}\left[\dfrac{\dfrac{dw}{dx}}{\dfrac{du}{dx}}\right]\dfrac{1}{\dfrac{du}{dx}} + 9(1+2w)\dfrac{\dfrac{dw}{dx}}{\dfrac{du}{dx}} - 4w^3 \\ \qquad\qquad\qquad\qquad - 6w^2 + 6w - 50 = 0, \\ -3\dfrac{\dfrac{dw}{dx}}{\dfrac{du}{dx}} + 3w^2 + 3w + 3 = \omega_8(x). \end{cases}
$$

If we use the second equation to simplify the first, we obtain the form

$$
(20)\quad \begin{cases} -9(1+2w)\dfrac{dw}{dx} + (14w^3 + 21w^2 + 33w - 41 \\ \quad -6w\,\omega_8(x) - 3\omega_8(x))\dfrac{du}{dx} + 3\omega_8' = 0, \\ 3\dfrac{dw}{dx} - (3w^2 + 3w + 3 - \omega_8(x))\dfrac{du}{dx} = 0. \end{cases}
$$

We have in (20) two differential equations of the first order with two dependent variables w and u and one independent variable x. There exist then two integrals involving the variables and two arbitrary constants.

From (17), ϕ can be determined by two quadratures, which brings the total number of arbitrary constants up to four. The variable u enters only through its derivative du/dx. Hence $u+k$, where k is an arbitrary constant, will satisfy the same system as u. By fixing arbitrarily the origin of the u-scale k may be made equal to zero. Thus we determine ϕ as a function of u except for three arbitrary constants.

If C be a conic, $\omega_8(x)=0$, and equations (18) reduce to the single equation

$$
(21)\quad -9\frac{d^2w}{du^2} + 9(1+2w)\frac{dw}{du} - 4w^3 - 6w^2 + 6w + 4 = 0,
$$

for which there exists an integral involving w as a function of u and two arbitrary constants. As in the preceding case ϕ may be determined as a function of u except for three arbitrary constants.

If we differentiate the third equation of (1′) with respect to v and eliminate y_u and y_{uv}, we obtain an equation in the form (14) where v is the independent variable. By means of this we study the projective properties of the curves $u=$const. and by choosing an arbitrary curve C' we can determine the function $\psi(v)$ except for three arbitrary constants.

Equations (13) show that the invariants A'' and B are determined except for six arbitrary constants. We shall now proceed to determine these six constants by choosing the straight lines of the Laplacian transforms.

Let $f(y_1, y_2, y_3)=0$ be the equation of the curve $C(v=v_0)$, where it is to be understood that y_1, y_2, and y_3 form a fundamental system of solutions of equations (1). Let $F(\rho_1, \rho_2, \rho_3)=0$ be the equation of the corresponding curve C_{-1}, generated by the point P_ρ, of the minus first Laplacian transform, where each ρ is defined as a function of y by the first of equations (4) which, on account of (11), become

$$(4') \qquad \rho = y_u - \frac{1-B}{3}y, \qquad \sigma = y_v - \frac{1-A''}{3}y.$$

$F(\rho_1, \rho_2, \rho_3)$ becomes a function of y_1, y_2, y_3, and B. According to the conditions set forth in this paper C_{-1} is a straight line, the degenerate minus first Laplacian transform. If we choose this line arbitrarily we will have two relations between the arbitrary constants in B.

Furthermore, equations (4′) show that the point P_ρ lies on the tangent to the curve C at P_y and the point P_σ on the tangent to C'. If we choose a fixed point $u=u_0$ on C the point P_ρ can occupy only a single infinity of positions. Since the line C_{-1} has been prescribed, the position of P_ρ will be determined, thus giving us a third relation between the arbitrary constants.

In a similar manner we may secure three relations between the constants by choosing a straight line C_1 to be the degenerate first Laplacian transform and a point $v=v_0$ on the curve C'. The six relations are sufficient to determine the six arbitrary constants. We may therefore state the following conclusion.

Choose two non-rectilinear but otherwise arbitrary analytic curves C and C' intersecting in a point P and having distinct tangents T and T' at P. Choose an arbitrary straight line C_{-1} intersecting T and another line C_1 intersecting T'. There exists one and only one net which contains the curves C and C' and which moreover has C_1 and C_{-1} for its degenerate first and minus first Laplacian transforms.

The family of curves $v=$const. may be obtained from the curve C by projective transformations. Similarly, the family $u=$const. may be obtained from C'.

THE UNIVERSITY OF OKLAHOMA

SOME PROPERTIES OF UPPER SEMI-CONTINUOUS COLLECTIONS OF BOUNDED CONTINUA*

BY W. A. WILSON

1. *Introduction.* If $T=\{t\}$ denotes a closed set of points and with each point t there is associated a unique bounded continuum X (or X_t) in such a way that (a) $X_t \cdot X_{t'}=0$ if $t\neq t'$, (b) at each point $t=\tau$ of T the upper closed limit of X_t as $t\to\tau$ is a part of X_τ, we say that $X=f(t)$ is an upper semi-continuous function in T. The collection of continua $\{X\}$ is also known as an upper semi-continuous collection of continua. These aggregates have been discussed by various writers here and abroad and enjoy numerous interesting properties.

R. L. Moore, in particular, has given an extensive treat-

* Presented to the Society, February 25, 1928.