In a similar manner we may secure three relations between the constants by choosing a straight line C_1 to be the degenerate first Laplacian transform and a point $v=v_0$ on the curve C'. The six relations are sufficient to determine the six arbitrary constants. We may therefore state the following conclusion.

Choose two non-rectilinear but otherwise arbitrary analytic curves C and C' intersecting in a point P and having distinct tangents T and T' at P. Choose an arbitrary straight line C_{-1} intersecting T and another line C_1 intersecting T'. There exists one and only one net which contains the curves C and C' and which moreover has C_1 and C_{-1} for its degenerate first and minus first Laplacian transforms.

The family of curves $v=$const. may be obtained from the curve C by projective transformations. Similarly, the family $u=$const. may be obtained from C'.

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SOME PROPERTIES OF UPPER SEMI-CONTINUOUS COLLECTIONS OF BOUNDED CONTINUA*

BY W. A. WILSON

1. *Introduction.* If $T=\{t\}$ denotes a closed set of points and with each point t there is associated a unique bounded continuum X (or X_t) in such a way that (a) $X_t \cdot X_{t'}=0$ if $t\neq t'$, (b) at each point $t=\tau$ of T the upper closed limit of X_t as $t\rightarrow\tau$ is a part of X_τ, we say that $X=f(t)$ is an upper semi-continuous function in T. The collection of continua $\{X\}$ is also known as an upper semi-continuous collection of continua. These aggregates have been discussed by various writers here and abroad and enjoy numerous interesting properties.

R. L. Moore, in particular, has given an extensive treat-

* Presented to the Society, February 25, 1928.

ment of the subject† and among other things has shown that, if the continua $\{X\}$ all lie in a plane but none of them cuts the plane, then the complement of $M=\Sigma[X]$ is a simply connected region whose frontier is a part of M if T is a simple arc and the complement of M is two simply connected regions whose frontiers are parts of M if T is a circumference. It is easily seen by examples that the frontiers of the complementary regions need not coincide with M in either case; it is the purpose of this note to give the conditions under which they do.‡

To this end we define $X=f(t)$ as a *minimal* upper semi-continuous function in T if there exists no upper semi-continuous function $Y=g(t)$ such that at every point t, $Y\subset X$, and at some point $Y\neq X$. If T denotes the interval $(-1\leqq t\leqq 1)$ and in some plane we let X_t be the point $(t, \sin(1/t))$ when $t\neq 0$ and the point set $x=0$, $-1\leqq y\leqq 2$ when $t=0$, then $X=f(t)$ is an upper semi-continuous function which is not a minimal function, but becomes so if we replace $X_0=f(0)$ by the set $x=0$, $-1\leqq y\leqq 1$. An example of a minimal upper semi-continuous function where no X is a point is given by the author in this Bulletin, vol. 32, p. 679.

2. *Notation.* The following notation will be convenient. If $X=f(t)$ in T and $M=\Sigma[X]$ we write $M=F(T)$. If T is a bounded continuum and $f(t)$ is upper semi-continuous, it is obvious that M is a bounded continuum; in this case we say that X is an *element* of M.

If T is a simple arc ab, $M=F(ab)$ will be called a generalized arc, or simply an arc if no confusion is caused. This may be denoted by X_aX_b and the elements X_a and X_b will be called the ends. Likewise, $M-(X_a+X_b)$ is

† R. L. Moore, *Concerning upper semi-continuous collections of continua*, Transactions of this Society, vol. 27, pp. 416–428.

‡ The attention of the reader is directed to an article by C. Kuratowski, *Sur la structure des frontières communes à deux régions*, Fundamenta Mathematicae, vol. 12 (1928), pp. 20–42, of which an advance copy was received while this paper was in press. Although Kuratowski's article is concerned chiefly with the converse problem, the reader will note a certain degree of similarity between the two papers.

called a (*generalized*) *open arc* and denoted by $X_a^*X_b^*$.

If T is a circumference C, $M=F(C)$ is called a (*generalized*) *simple closed curve*. Obviously any two elements X_1 and X_2 divide M into two arcs having X_1 and X_2 as end elements and no other common points.

The plane will be denoted throughout by Z.

3. *Certain Corollaries.* Certain properties of the sets under consideration are either corollaries of Moore's work or are so easily demonstrated that their proofs are omitted.

(a) *If $M=F(ab)$ is a generalized arc in a plane Z, no sub-continuum of M separates X_a from X_b unless some element of M does.*

(b) *If $M=F(C)$ is a generalized simple closed curve in a plane, no two elements of M are separated by a sub-continuum of M.*

These are readily proved with the aid of a theorem of Janiszewski.†

(c) *In a plane Z let $M=F(ab)$ be a generalized arc and no element of M separate X_a from X_b, or let $M=F(C)$ be a generalized simple closed curve. Then there is not more than one element X such that a bounded component of $Z-X$ contains $M-X$.*

(d) *In a plane Z let $M=F(ab)$ be a generalized arc and no element of M separate X_a from X_b, or let $M=F(C)$ be a generalized simple closed curve. For each element X let the component of $Z-X$ containing $M-X$ be unbounded. Let Y be the union of X and the components of $Z-X$ containing no points of M. Then $Y=g(t)$ is upper semi-continuous in ab or C, respectively.*

4. LEMMA. *Let $M=F(C)$ be a generalized simple closed curve in a plane Z. Then $Z-M$ has two components whose frontiers have points in every element of M and every other component has a frontier which is a part of some element.*

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† Z. Janiszewski, *Sur les coupures du plan faites par des continus*, Prace Matematyczno-Fizyczne, vol. 26, Theorem A. See also R. L. Moore, *Concerning the prime parts of certain continua which separate the plane*, Proceedings of the National Academy of Sciences, vol. 10, p. 173.

PROOF. This is really a corollary of R. L. Moore's work. By §3 (c) there is at most one element X such that a bounded component of $Z-X$ contains $M-X$. Since inversion with respect to a point within this component will make its image unbounded, there is no loss in generality if we assume that for every element X the unbounded component of $Z-X$ contains $M-X$.

Defining $Y=g(t)$ as in §3 (d), it follows from this reference that $N=\Sigma[Y]$ is a generalized simple closed curve no element of which separates Z. In this case Moore has shown (loc. cit., Theorem 11) that $Z-N$ consists of two components R and S, and the frontier of each of these has at least one point on every element Y. It is readily seen that these are also components of $Z-M$. Hence the first part of the lemma is proved.

That the frontier of each of the other components of $Z-M$ is a part of some element X is a consequence of the definition of $Y=g(t)$.

DEFINITION. The components of $Z-M$ which have frontier points on every element of M will be called *principal* components.

COROLLARY. *Let $C=\{t\}$ be a circumference, let $X=f(t)$ be an upper semi-continuous function defined over C, and let $M=\Sigma[X]$ lie in the plane Z. If M is the common frontier of two components of $Z-M$, then $f(t)$ is a minimal upper semi-continuous function.*

PROOF. Let R and S be the components of $Z-M$ having the frontier M. If the theorem is not true, let $Y=g(t)$ be upper semi-continuous over C, let $Y\subset X$ for every t, and let $Y\neq X$ for some t. Let $N=\Sigma[Y]$.

By the above lemma, $Z-N$ has two principal components R' and S'. Since M is an irreducible cut of Z between points of R and S and N is a proper part of M, R and S lie in the same component of $Z-N$, and this must contain all the points of $M-N$. Suppose that $S'\cdot(R+S)=0$. Then the frontier of S' is a part of some element of M, as S' is a component of $Z-M$. This is a contradiction, since N contains points of every element of M.

5. LEMMA. *Let $M=F(C)$ be a generalized simple closed curve in a plane Z, and let R be one of the principal components of $Z-M$. Let a and b be points of M accessible from R and lying on different elements A and B of M. Let A and B divide M into the arcs M_1 and M_2. If F is the frontier of R, $F=H+K$, where H and K are sub-continua of M_1 and M_2, respectively, joining a and b.*

PROOF. Let m be a point of R and ma and mb be simple arcs lying in R except for the points a and b and having only m in common. R. L. Moore has shown (loc. cit., p. 423) that the arc $ab=ma+mb$ divides R into two simply connected regions R_1 and R_2 such that their frontiers are parts of $M_1+a^*b^*$ and $M_2+a^*b^*$, respectively.

Let H and K, respectively, denote those points of these frontiers not on a^*b^*; then $H\subset M_1$, $K\subset M_2$. Since the frontier of R_1 is a continuum, every point of H can be joined to a or b by a sub-continuum of H. If H is not a continuum, $H=H_1+H_2$, where H_1 and H_2 are continua containing a and b, respectively, and $H_1\cdot H_2=0$. This hypothesis would give a contradiction by the theorem of Janiszewski referred to earlier, for neither ab, H_1, nor H_2 separates R_1 from R_2, while $(ab)\cdot H_1=a$, $(ab)\cdot H_2=b$, and $(ab+H_1)\cdot(ab+H_2)=ab$. Thus H, and in like manner K, is a continuum.

Now $F\supset H+K$. On the other hand every frontier point of R is necessarily one of R_1, or of R_2, or of both. Hence $F=H+K$.

6. THEOREM. *Let $C=\{t\}$ be a circumference, let $X=f(t)$ be a minimal upper semi-continuous function defined over C, and let $M=\Sigma[X]$ lie in the plane Z. Then M is the frontier of two components of $Z-M$ and the frontier of each of the remaining components is a part of some element of M.*

PROOF. The last assertion is a restatement of a previous result. (See §4.) Let R and S be the principal components of $Z-M$, and let F be the frontier of one of them, say R. Since accessible points are everywhere dense in F and F contains at least one point in every element X, there is an everywhere dense set of points t each of whose corresponding elements contains an accessible point of F.

Orient the points of C, let τ be a fixed point t, and let $\{t_i\}$ and $\{t_i'\}$ be sequences of points $\{t\}$ such that $t_1 < t_2 < \cdots \to \tau$, $t_1' > t_2' > \cdots \to \tau$, and for each t_i and t_i' the corresponding X_i or X_i' contains an accessible point of F.

Let M_i be the arc of M joining X_i and X_i' and containing $X_\tau = f(\tau)$. Obviously $X_\tau = \Pi_1^\infty [M_i]$. By the previous lemma M_i contains a sub-continuum F_i of F joining X_i and X_i', and $(F - F_i) \cdot X_\tau = 0$. Moreover, $\Pi_1^\infty [F_i]$ is a continuum.

But $F_i \subset M_i$; hence $\Pi[F_i] \subset X_\tau$. As $(F - F_i)\ X_\tau = 0$, $F \cdot X_\tau \subset F_i$, whence $F \cdot X_\tau = \Pi[F_i]$. Thus we have shown that for each t, $Y_t = F \cdot X_t$ is a continuum.

On the other hand, $\overline{\lim}_{t \to \tau}\ Y_t \subset F \cdot X_\tau \subset Y_\tau$. For $Y_t \subset F$ $Y_t \subset X_t$, and $X_t = f(t)$ is upper semi-continuous. Thus $Y_t = g(t)$ is upper semi-continuous. But $f(t)$ is a minimal upper semi-continuous function. Hence for every t, $Y_t = X_t$ and so $F = M$.

Corollary. *Let M satisfy the hypotheses of the above theorem, let A and B be any two elements of M, and M_1 and M_2 the complementary arcs of M thus determined. Then $M = H_1 + H_2$, where $H_1 \subset M_1$, $H_2 \subset M_2$, $H_1 \cdot H_2 = \alpha + \beta$, where $\alpha \subset A$, $\beta \subset B$, and both H_1 and H_2 are continua irreducible between α and β.*

Proof. Let $M = H_1 + H_2$ be an irreducible decomposition* of M such that $H_1 \subset M_1$ and $H_2 \subset M_2$. Then $H_1 \supset M_1 - (A + B)$ and $H_2 \supset M_2 - (A + B)$. By the above theorem M is an irreducible cut of the plane between a point of R and one of S. By a theorem proved elsewhere† H_1 and H_2 are both irreducible between $\alpha = A \cdot H_1 \cdot H_2$ and $\beta = B \cdot H_1 \cdot H_2$.

* The decomposition $M = H_1 + H_2$ is called irreducible if H_1 and H_2 are continua and there exists no proper sub-continuum K of H_1 or H_2 such that $M = K + H_2$ or $M = H_1 + K$, respectively.

† W. A. Wilson, *On irreducible cuts of the plane between two points*, Annals of Mathematics, vol. 29, §9.

7. LEMMA. *Let $M=F(ab)$ be a generalized arc lying in a plane Z and let no element of M separate X_a from X_b. Then $Z-M$ has one component whose frontier has points on every element of M and every other component has a frontier which is a part of some elements.*

PROOF. This is a corollary of a theorem by R. L. Moore (loc. cit., Theorem 9). It is proved in the same manner as the lemma of §4.

8. THEOREM. *Let the aggregate $\{t\}$ be a simple arc ab, let $X=f(t)$ be a minimal upper semi-continuous function defined over ab, and let $M=\Sigma[X]$ lie in the plane Z, while no element of M separates X_a from X_b. Then M is a continuum irreducible between X_a and X_b.*

PROOF. By §7 there is one component R of $Z-M$ which has frontier points on every element of M. Since accessible points are everywhere dense, there is a decreasing sequence $\{t_i\}$ where $t_i \to a$ and an increasing sequence $\{t_i'\}$ where $t_i' \to b$, such that $X_i=f(t_i)$ and $X_i'=f(t_i')$ contain accessible points x_i and x_i', respectively.

Let x_ix_i' be a simple arc lying in R except for the end points. Let u run over a circumference C. Let the segment t_it_i' of ab be homeomorphic with an arc cd of C in such a way that t_i corresponds to c and t_i' to d. Let the arc x_ix_i' be homeomorphic to the complementary arc $dc=\overline{C-cd}$ in such a way that x_i corresponds to c and x_i' to d. Now define the function $Y=g(u)$ as follows. If $u=c$, Y is a sub-continuum of X_i irreducible about x_i and $\overline{\lim}\, X_t$ as $t \to t_i$ in t_it_i'; if $u=d$, Y is a sub-continuum of X_i' irreducible about x_i' and $\overline{\lim}\, X_t$ as $t \to t_i'$ in t_it_i'; if u is any other point of cd and t is the corresponding point of t_it_i', $Y=X_t=f(t)$; if u is a point of $dc-(c+d)$, Y is the corresponding point x of the simple arc x_ix_i'. It is readily seen that $Y=g(u)$ is a minimal upper semi-continuous function and that $N=\Sigma[Y]=G(C)$ is a generalized simple closed curve.

Now Y_c and Y_d determine two generalized arcs $N_1=G(dc)$ and $N_2=G(cd)$ having these elements as ends and no other common points. If we set H_1 equal to the simple arc x_ix_i' and $H_2=N-x_ix_i'+x_i+x_i'$, it is evident that $H_1\subset N_1$, $H_2\subset N_2$, and that $N=H_1+H_2$ is an irreducible partition of N. Then by §6, Corollary, H_2 is irreducible between x_i and x_i'. As $H_2\supset N_2-(Y_c+Y_d)$, $H_2\supset X_iX_i'-(X_i+X_i')$. That is, any sub-continuum of the arc X_iX_i' joining the end elements contains all elements between them. Hence any sub-continuum of M joining X_a and X_b contains every point of all the elements between X_i and X_i'. As $t_i\rightarrow a$ and $t_i'\rightarrow b$, this means that it contains every point of $M-(X_a+X_b)$. But, since $X=f(t)$ is a minimal upper semi-continuous function, $M=\overline{M-(X_a+X_b)}$. Hence M is irreducible between X_a and X_b.

COROLLARY. *Let the aggregate* $\{t\}$ *be a simple arc ab, let* $X=f(t)$ *be a minimal upper semi-continuous function defined over ab, and let* $M=\Sigma[X]$ *lie in the plane Z, while no element of M separates* X_a *from* X_b. *Then M is the frontier of one component of* $Z-M$ *and the frontier of each of the remaining components is a part of some element of M.*

PROOF. By §7 there is one component of $Z-M$ whose frontier has points on every element of M. Since this frontier is a continuum joining X_a and X_b and is a part of M, it must coincide with M by the above theorem. The last part of the corollary is merely restated from §7.

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