RATIONAL QUINTIC SURFACES WITH TWO SKEW DOUBLE LINES*

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The purpose of this paper is to obtain by different methods, and where possible to extend, the results given by Clebsch and Hill† for quintic surfaces having two distinct skew double lines. In a second paper, which will appear in an early issue of this Bulletin, I discuss a surface, due to Montesano, which has two consecutive skew double lines.

The equation of a quintic surface that has xy and zw for double lines, but not further specialized, contains twenty-four terms. If we take a point on each of them, say $(0:0:z_1:1)$ and $(x_1:1:0:0)$ we find that for a given surface there are thirteen pairs of values of x_1 and z_1 such that the line joining corresponding points lies in the surface. If we allow the coefficients to vary we find that eleven of these transversals may be assigned arbitrarily and the remaining two are then determined. That is, there is a single infinity of such surfaces all of which have the eleven transversals and the other two determined by them. The form of the equation shows that there are six pinch points on each of the double lines.

The surface is obviously rational, that is, its points may be put in one to one correspondence with the points of a plane. For the line drawn from any point P of the surface to intersect the two double lines meets a given plane, π_0, in a point P', which we may consider the image of P. A section of the quintic surface has of course two double points where its plane cuts the double lines. Therefore, establishing the correspondence in the way just indicated, as P moves on the plane section the surface generated by the

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† Clebsch, Mathematische Annalen, vol. 1, p. 253.
Hill, Mathematical Review, vol. 1, p. 1.

line $P'P$ is of degree $2\times 1\times 1\times 5-2\times 1-2\times 1$, or 6. And the two lines, a and b, double on the quintic, are triple on this sextic surface. Hence the images of the plane sections are sextic curves having in common two triple points where the lines a and b meet π_0, and thirteen simple points where the transversals meet it. If we did not know the number of the latter, we could infer it to be thirteen, since two sextics of the system can have only five variable intersections. As mentioned above, eleven of these transversals determine the other two, and there is a pencil of surfaces having the two double lines and the thirteen transversals. But the ruled sextic determined as above by any plane section of a quintic of the pencil meets the other quintics also in plane quintic curves. Thus we have ∞^3 and not ∞^4 such ruled sextics. Hence we have the following theorem.

THEOREM 1. *Of the ∞^4 plane sextics having in common two triple points and eleven simple points there are ∞^3 (instead of ∞^2) that pass also through two certain points determined by the given thirteen.*

We see how this determination is effected when we consider the images of the double lines. From each point of one double line we must draw the two tangents to the surface, one in each sheet, that meet the other. Hence the two images of a point of a double line lie on a line through the point in which the other double line meets π_0. Let the double lines a and b meet π_0 at B and A respectively. Since any plane through a double line is tangent to the surface where the residual cubic meets the double line, the image of a has a double point at B and a triple point at A, and is met by any line through B in three points and by any line through A in two (associated) points. Thus the image of a double line is a quintic which has a double point where the double line itself meets π_0, and a triple point where the other double line meets it, and which contains the points of intersection of the thirteen transversals with π_0. But a quintic that has a fixed triple point and a fixed double point is fully determined by 11 other simple points; and two such quintics have just two more intersections, which are the points we seek. This gives the following simple but striking result. In a plane, π_0, take two points A and B and 11 other points C. Let the two remaining intersections of the two quintics both of which contain the 11 points C, and which have respecitvely a triple point at A and a double point at B, and vice versa, be P and Q. Then if any two skew lines neither of which lies in π_0, be drawn through A and B, there will be a pencil of quintic surfaces having the two skew lines for double lines and containing the 13 transversals determined by the points C and P and Q.

The configuration in π_0 may be simplified by a quadratic transformation. Taking for fundamental triangle the points A and B in which the double lines meet the plane, and C, where any one of the transversals meets it, we see immediately that the sextics which are the images of the plane sections become quintics having double points at A and B, and passing through the twelve points that correspond to the points in which the other twelve transversals meet π_0. The images of the double lines become quartics, one having a double point at A and a simple point at B, and the other a node at B and a simple point at A, and both passing through the twelve new points. C goes into the line AB which is now the image of the line on the surface that originally corresponded to C. Evidently the point C can be chosen in thirteen ways. That is, the original plane system of sextics can be reduced to the system of quintics having in common two double points and twelve simple points in thirteen ways. This was noted by Clebsch. Referring now to the new system let l be the line on the surface whose image is AB. To sections of the surface by planes through l correspond quintics each composed of the line AB and a variable quartic through A and B and the other twelve base points. These quartics have only two variable intersections with AB, since l meets both double lines of the surface. It was seen above that the image of a double line is a quartic, having a double point at A (or B) and a simple point at B (or A), and passing through the twelve simple base points. Hence the images of the double lines are special curves of the pencil of quartics that correspond to the plane sections through l; and the fact that they have double points at A and B shows that any two curves of the pencil have two intersections at each of these points. This may be restated as follows.

THEOREM 2. *The fourteen fundamental points are the base points of a pencil of quartics that have a common tangent at two of them.*

On the image of a double line the points are paired, since to a point on a double line must correspond two points in the plane. Thus the quintic corresponding to any plane section meets the image of a double line in two (associated) points aside from the fundamental points. Two quartics of the pencil are tangent to AB. The corresponding plane quartics on the surface are tangent to l where it is tangent to the parabolic curve, and their planes are doubly stationary, that is, they are stationary planes in the developable of the stationary tangent planes. Clebsch remarks that there are twenty-five quartics of the plane pencil that have a node aside from A or B. The number would seem to be twenty-three, since the two special quartics of the pencil (the images of the double lines) count for two each. That is, $3(n-1)^2-4=23$ for $n=4$. Thus if we consider a pencil of cubics that have a common tangent there are only ten $(3\times 2^2-2)$ that have a node aside from the one whose node is at the point of tangency. Thus the plane sections through l (and hence any transversal) are in general doubly tangent to the surface, but twenty-three of them are triply tangent; and evidently two, that is, the planes determined by the transversal and the double lines, cut the surface in conics, and are quadruply tangent.

In the above the plane system has been derived by geometrical projection and quadratic transformation. If we start with the statement that the plane system consists of the quintics having in common two double points, A and B, and twelve simple points, we immediately see that there must be a relation among the base points, for otherwise they would determine only ∞^2 quintics. We may, however, easily find this relation by means of the theory of rational surfaces as developed by Caporali.* It tells us that the complete image of the double lines is a curve of degree eight which must consist of the two quartics described above. Considering them in connection with the pencil of quartics, the images of the plane sections through l, we find as before the relation connecting the base points.

Some other properties of the surface will now be developed. Let a be the double line whose image is the quartic with double point at A. To the quintic composed of this quartic and the line AB corresponds the section of the surface whose plane contains a and l. Therefore the point A must correspond to the conic cut from the surface by that plane. All directions at A correspond to points on this conic. The directions of the tangents of the image of a correspond to the points at which the conic meets a. To the direction AB corresponds the intersection of the conic with l which does not take place on the other double line, b. All sections of the surface by planes through l, including the conic in question, pass through the point where l meets b; and corresponding to that point we have at A the common direction of the pencil of quartics which are the images of those sections. The image of b has at A this same direction. Since b is a double line, its intersection with l corresponds to another point in the plane, which must lie on AB, and which is the fourth point in which the image of b meets AB, in addition to the double point at B and the simple intersection at A. Thus this pair of associated points on the image of b lie on a line, BA, through the double point B. In an exactly similar

* *Sistemi Triplimente Infiniti*, Collectanea Mathematica, Hoepli, 1881.

way all directions at B correspond to points on the conic whose plane contains l and the double line b.

It is a distinctive feature of this correspondence that any ray through the double point, A or B, of the quartic which is the image of a double line meets the curve in two associated points, that is, points which correspond to a single point of the surface. For the image of b and any line through A make up a quintic of the system, the corresponding section of the surface being a plane cubic whose plane contains b. This plane meets a in a point whose two images must lie on the line through A. Through any point of the surface there are two such plane cubics corresponding to rays through A and B in the plane. The points of a pair corresponding to a point of the double curve are such that the net of fundamental curves that pass through one pass also through the other. If two such points come together they are the image of a pinch point of the surface. Such points lie on the Jacobian of any net of fundamental curves. In our case, since we can draw six tangents to a uninodal quartic from its double point, there are six pinch points on each of the double lines. Again, from A, which is a simple point on the image of b, eight tangents can be drawn to that quartic. To these tangents correspond plane cubics on the surface which are tangent to b where it is tangent to the parabolic curve, and whose planes are stationary planes in the developable of the stationary tangent planes. There are of course eight such sections that contain a. To each of the twelve simple base points corresponds a line that meets both double lines. To a ray through A (or B) and one of these twelve points corresponds the conic whose plane contains b (or a) and the line corresponding to that base point. Thus there are twenty-six such conics, including the two whose planes pass through l. To the line joining any two simple base points, or to the conic through A and B and three simple base points, corresponds a skew cubic. There are thus $66+220$, or 286 of these. If two simple base points lie on a line through A the corresponding lines on the surface will meet on a, and their plane, which contains b, meets the surface in a residual line, whose image is the ray joining A to the base points. That is, two of the twenty-six conics just mentioned have become line pairs. Clebsch has pointed out that the ∞^3 hyperboloids which contain the double lines meet the surface in residual skew sextics whose images are the conics through A and B. The ∞^2 hyperboloids that contain also the line l intersect the surface in residual skew quintics that have for their images the line AB and the ∞^2 lines of the plane.

The genus of a curve of order n on the surface does not exceed the greatest integer in $(2n^2+2n+33)/20$. Let m be the order of the corresponding curve in the plane, and let it have points of multiplicity $\alpha_1, \alpha_2, \cdots, \alpha_{12}$ at the simple base points, and of multiplicity β_1 and β_2 at A and B. Then we have $5m-[\Sigma\alpha+2\Sigma\beta]=n$. Letting $\Sigma\alpha+2\Sigma\beta=k$, we have $m=(n+k)/5$. Then the genus of the plane curve, which is the same as that of the corresponding curve on the surface, is

$$\frac{(n+k-5)(n+k-10)}{50}-\frac{1}{2}\left[\sum\alpha(\alpha-1)+\sum\beta(\beta-1)\right].$$

Differentiating with respect to the twelve α's and two β's, we have fourteen equations from which it follows easily that for a maximum genus $k=4n+10$, the α's are all equal to $(n+3)/5$, and $\beta_1=\beta_2=(4n+7)/10$. For these values the expression for the genus becomes $(2n^2+2n+33)/20$.

The characteristic numbers of the tangent cone are obtained without difficulty by the method applicable to rational surfaces, or by the general formulas of Salmon and Plücker. For the sake of comparison with the forms of the surface to be treated later we may note that the order of the tangent cone is 16. Its class, or class of the surface, is 34. Its genus is 23; and it has 42 cuspidal edges, and 40 double edges.

In concluding this paper we may consider some special groupings of the base points. The general methods of producing nodes on unicursal surfaces are to allow two or more base points to coincide, or to place on a line a number of base points equal to the order of the fundamental curves. Thus, if two simple base points coincide we get an ordinary conic node. If three become consecutive, but not collinear, the result is a binode, B_3, that reduces the class by three, etc. In our case the fourteen fundamental points are the common intersections of a pencil of quartics that have a common tangent at two of them. Therefore, when we have chosen these two, say A and B, and the directions thereat, we have only nine at our disposal. The remaining three which are thus determined may be called residual points. We can not put five of the nine on a line, nor any of them on the line AB. But we can put four of them on a line, and we then get on the surface a fourteenth line, m, which meets l and the lines corresponding to the four base points, but which does not meet the double lines. To sections of the surface by planes through m correspond a pencil of quartics having double points at A and B, and passing through the five remaining points (of the nine) and the three residual points. An arbitrary quartic of the pencil meets the line containing the collinear base points four times, and thus the corresponding plane is quadruply tangent to the surface. But thirteen quartics of the pencil have a third node and their corresponding planes are quintuply tangent. If these ten points were arbitrary there would be but one quartic having double points at two of them and passing through the other eight. The section that contains m and l meets the double lines where l does. Its image in the plane, therefore, in addition to the line containing the four base points, is the line AB, and a cubic which has the common directions at A and B, and passes through the other eight points; thus satisfying twelve conditions. If we now put four of the five points still at our disposal on a line we get a fifteenth line, n, on the surface, which meets l and m and four others, but not the double lines. Clebsch remarked that no line can meet more than five of the thirteen transversals to the double lines. The residual section by the plane of l, m, n is a conic whose image in the plane is also a conic, which has the common directions at A and B and passes through the four other points, thus fulfilling eight conditions. Thus, if four points of such a set of fourteen are collinear, or if eight of them lie in sets of four on two lines, the remaining points are related in the manner just noted. If we arranged the nine points in a triangle, one at each vertex and two on each side, we should get three skew lines on the surface, each meeting l and four other lines, but not meeting the double lines.

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A NOTE ON THE RATIONAL PLANE QUARTIC CURVE WITH CUSPS OR UNDULATIONS*

BY J. H. NEELLEY

1. *Introduction.* In a recent paper† compound singularities of the rational plane quartic curve have been considered, but cusps and undulations were not incorporated in that article because of their widespread discussion by other writers. However, the errors or ambiguities in previous treatments of these two singularities are cleared up by this paper.

2. *A System of Cusp Invariants.* It is well known that all types of the rational plane quartic curve with simple singularities are given to within projection by plane sections of the Steiner Romische Surface S_3^4 of order three and class four. When referred to its tropes S_3^4 has the equation

$$(x_0)^{1/2} \pm (x_1)^{1/2} \pm (x_2)^{1/2} \pm (x_3)^{1/2} = 0. \tag{1}$$

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†J. H. Neelley, *Compound singularities of the rational plane quartic curve*, American Journal of Mathematics, vol. 49 (1927), pp. 389-400.