ON CONNECTED AND REGULAR POINT SETS*

BY R. L. WILDER

In a recent number of this Bulletin† G. T. Whyburn showed that if A and B are any two points of a connected and regular point set M, and K denotes the set of all points of M which separate‡ A and B in M, then $K+A+B$ is a closed and bounded point set. In the present paper I shall show that this theorem is susceptible of quite simple proof, and admits of two obvious generalizations which hold in space of n dimensions. The methods used in proving these generalizations are also employed to show that if N is a closed and bounded point set which lies in a connected subset of an open subset, F, of a connected and regular point set, M, then F contains a bounded, connected and regular set which contains N. These results are then applied to give certain theorems concerning continuous curves.

Use will be made of the notions *region of M* and *simple chain of regions* as introduced in my paper, *The non-existence of a certain type of regular point set*,§ as well as of Theorem 1 of that paper; these extend readily to n-dimensional space, if in the definition of *region of M* "circle" is replaced by "sphere." Furthermore, since the Borel property is employed I make note of the following lemma, the proof of which should be quite obvious.||

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† G. T. Whyburn, *Concerning connected and regular point sets*, this Bulletin, vol. 33 (1927), pp. 685–689.

‡ If A, B and X are points of a connected set M, then X is said to separate A and B in M if $M-X$ is the sum of two mutually separated point sets which contain A and B respectively.

§ This Bulletin, vol. 33 (1927), pp. 439–446. This paper will be referred to hereafter as N. E. R.

|| It will be understood hereafter without explicit statement that the results hold for sets imbedded in euclidean space of n dimensions.

LEMMA. *If M is a connected and regular point set, and N is a closed and bounded subset of M, and G is a collection of regions of M which cover N, then G contains a finite subset which covers N.*

(By virtue of the property of regularity, there exists, concentric with the sphere used in determining a region R of M, a sphere S such that *all* points of M interior to S are also points of R. When it is said, then, that a collection of regions, G, *covers* a subset N of M, it is meant that every point of N is within the sphere S which corresponds to some region of the collection G.)

DEFINITION. If N is a subset of a connected set M and P is a point of $M-N$, then P is said to *separate N in M* if $M-P$ is the sum of two mutually separated sets each of which contains at least one point of N.

THEOREM 1. *If N is a closed and bounded subset of a connected and regular point set M and K denotes the set of all points which separate N in M, then $K+N$ is a closed and bounded point set.*

PROOF. Suppose the set $K+N$ is not closed. Then there is a point P which is a limit point of this set and which does not belong to it. As N is closed, P is a limit point of K, but not of N. If x is any point of M, distinct from P in case P is a point of M, there exists a region of M which covers x and which neither contains P nor has P as a limit point. Let G denote the collection of all such regions. By the above lemma, G contains a finite set of regions, R_1, $R_2, \cdots, R_n$, such that every point of N is in at least one of these regions. For every i ($i=1, 2, \cdots, n$), let P_i be a point of N in R_i. For every two points P_i and P_{i+1} ($i=1, 2, \cdots, n-1$) there exists, by Theorem 1 of N. E. R., and by virtue of the fact that P does not separate these points in M,* a

* Use is made in this connection and in the proof of Theorem 2 below, of Lemma 2 of my paper *A characteristization of continuous curves by a property of their open subsets*, Fundamenta Mathematicae, vol. 11(1928), pp. 127–131.

simple chain of regions of G from P_i to P_{i+1}. The regions $R_1, R_2, \ldots, R_n$ together with the regions which go to make up these simple chains form a finite set, g, of regions of G, and if H denotes the set of all points of M contained in regions of g, then H is a connected and bounded set. As P is not a limit point of H there exist in the vicinity of P points of K which are not in H. But clearly such points cannot separate N in M since N is a subset of H. Thus the supposition that $K+N$ is not closed leads to a contradiction.

To show that $K+N$ is bounded, let G be any collection of regions of M covering N and proceed as above to establish the existence of a bounded connected set, H, which is a subset of M and contains N. As every point of K must lie in H, it is clear that $K+N$ is bounded.

DEFINITION. If A and B are any two distinct subsets of a connected set M, and X is a point of $M-(A+B)$, then X is said to *separate A from B in M* if $M-X$ is the sum of two mutually separated sets which contain A and B, respectively.

THEOREM 2. *If A and B are any two distinct subsets of a connected and regular point set M, such that $A+B$ is closed and bounded, and K denotes the set of all points which separate A from B in M, then $K+A+B$ is a closed and bounded set.*

INDICATION OF PROOF. Select P and G as in Theorem 1. As P does not separate A from B in M, there is a connected subset of $M-P$ which contains a point P_1 of A and a point P_2 of B. Hence from G can be selected a simple chain from P_1 to P_2.

The corollarys of Theorems 1 and 2 applied to continuous curves* are obvious.

As a direct consequence of Theorems 1 and 2 and of the fact that the difference of two closed sets is both an F_σ and a G_δ,† we have the following theorem.

* In this paper a continuous curve is considered as a closed, connected and regular point set not necessarily bounded.

† See F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig, 1914, p. 306. An F_σ is the sum of a denumerable set of closed sets and a G_δ is the set of points common to a denumerable set of open sets.

THEOREM 3. *Let M be a connected and regular point set. Then if N is any closed and bounded subset of M, the set of all points which separate N in M is both an F_σ and a G_δ: and if A and B are any two distinct subsets of M such that $A+B$ is closed and bounded, the set of all points which separate A from B in M is both an F_σ and a G_δ.**

By methods similar to those used in proving Theorem 1 and noting in addition the fact that a region in a connected and regular point set is itself a regular set (Theorem 2 of N. E. R.), we have the following result.

THEOREM 4. *If K is a closed subset of a connected and regular point set M and N is a closed and bounded subset of a connected subset of $M-K$, then N lies in a bounded, connected and regular subset of $M-K$ which has no limit point in K.*

The analog of Theorem 4 for continuous curves may be stated as follows.

THEOREM 5. *If N is a closed and bounded subset of an open subset, Q, of a continuous curve M, and N lies in some connected subset of Q, then N lies in a bounded continuous curve which is a subset of Q.*

PROOF. Hahn has shown† that if P is any point of M and r is any positive number, there exists a continuous curve $M(P, r)$ which is a subset of M, contains every point of M less than a certain distance d (dependent on r) from P, and is such that all of its points are at a distance less than r from P. The set of all points $\{x\}$ of M such that x is joined to P by a connected subset of M every point of which is at a distance less than d from P constitutes a region of M and

* In this connection it may be of interest to note that it has been shown that the set of all cut points of a continuous curve is an F_σ. See C. Zarankiewicz, *Sur les points de division dans les ensembles connexes*, Fundamenta Mathematicae, vol. 9 (1927), pp. 124–171, Theorem 17.

† H. Hahn, *Mengentheoretische Charakterisierung der stetigen Kurve*, Wiener Akademie Sitzungsberichte, vol. 123, Part IIa, pp. 2433–2489; see Theorem XXI, p. 2475. Although Hahn states his result for a bounded regular continuum, it is clear that it holds for any regular continuum.

will be denoted by $M'(P, r)$. Clearly $M'(P, r)$ is a subset of $M(P, r)$.

If P is any point of Q, let r be a positive number less than the distance* from P to $M-Q$. Then the sets $M(P, r)$ and $M'(P, r)$ are subsets of Q. By a method similar to that used in proving Theorem 1 it can be shown that there exists a connected set H' which consists of a finite number of the regions of type $M'(P, r)$ and contains N. The set H composed of the sets $M(P, r)$ associated with those sets $M'(P, r)$ which constitute H' is a bounded continuous curve lying in Q and containing all points of N.

Theorem 5 is a generalization of a result obtained by R. L. Moore† to the effect that if Q is an open subset of a continuous curve and A and B are two points which lie in a connected subset of Q, then A and B are joined by a simple continuous arc which lies wholly in Q. As I have shown elsewhere‡ that this property is sufficient that a continuum be a continuous curve, it follows that the property stated in Theorem 5 also serves to characterize a continuous curve.

As I have indicated in a recent paper,§ subsets of a point set M may be separated in M in different senses. For our present purposes we employ the following definitions.

DEFINITION. If N is a subset of a connected set M and P is a point of $M-N$, then P is said to *separate N in M in the weak sense* if there exist two points of N which do not lie in a connected subset of $M-P$.

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* That is, the greatest lower bound of all distances Px, where x is a point of $M-Q$.

† *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), pp. 254–260, Theorem 1.

‡ *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), pp. 340–377, Theorem 18.

§ *A characterization of continuous curves by a property of their open subsets*, Fundamenta Mathematicae, vol. 11 (1928), pp. 127–131. The terminology "*separates* in the strong (or weak) sense" should not be confused with "*disconnects* in the strong (or weak) sense" as introduced by R. L. Moore in *Concerning the cut-points of continuous curves and of other closed and connected point-sets*, Proceedings of the National Academy of Sciences, vol. 9 (1923), pp. 101–106.

DEFINITION. If A and B are any two distinct subsets of a connected set M and X is a point of $M-(A+B)$, then X is said to *separate A from B in M in the weak sense* if $M-X$ contains no connected subset which contains points of both A and B.

As examples of cases where these two definitions of "separate" are satisfied, but where the preceding definitions* are not satisfied, consider the following examples.

Examples. Let M_n $(n=1, 2, 3, \cdots)$ denote the straight line interval joining the points $(0, 0)$ and $(1, 1/n)$. Also, denote the points $(1/2, 0)$, $(1, 0)$ and $(0, 0)$ by A, B, and X, respectively, and let $M=A+B+\sum_{n=1}^{\infty}M_n$.

The set M is connected, but the set $M-X$ contains no connected set containing A and B. Hence X separates A from B in the weak sense. However, there is no separation of $M-X$ into two mutually separated subsets containing A and B, respectively, and thus X does not separate A from B in M in the strong sense, that is, in the sense of the definition which immediately precedes Theorem 2.

If we let N denote the set of all points with rational coordinates in the interval $[0, 1]$ of the X axis, except $(0, 0)$, and define M_n and X as above, and let M now denote the set $N+\sum_{n=1}^{\infty}M_n$, it is easy to see that X separates N in M in the weak sense, but not in the strong sense, i.e., in the sense of the definition preceding Theorem 1.

On the basis of Lemma 2 of my paper *A characterization of continuous curves by a property of their open subsets*† we have the following extension of Theorems 1–5.

THEOREM 6. (1) *Theorems* 1, 2 *and* 3 *still hold true if "separate" be interpreted to mean "separate in the weak sense"*; (2) *Theorem* 4 *still holds true if the words "N is a closed and bounded subset of a connected subset of $M-K$" be replaced by "N is a closed and bounded subset of $M-K$ such that there is no separation of $M-K$ into two mutually separated sets each*

* That is, the definitions preceding Theorems 1 and 2, respectively.

† Loc. cit.

of which contains points of N;" and (3) *Theorem* 5 *still holds true if the words "N lies in some connected subset of Q" be replaced by "there is no separation of Q into two mutually separated sets each of which contains points of N."*

In conclusion we may note the following application of part (1) of Theorem 6 to the theory of irreducibly connected sets.*

THEOREM 7. *If the connected and regular point set M is irreducibly connected about a closed and bounded set N, then M is a bounded continuous curve.*

PROOF. Let P be any point of $M-N$. Then $M-P$ contains no connected subset which contains N, since M is irreducibly connected about N. That is, $M-N$ is the set of points of M which separate N in M in the weak sense, and accordingly by Theorem 6, part (1), the set $(M-N)+N=M$ is bounded and closed. Hence M is a bounded continuous curve.

It may be pointed out that Whyburn's Theorem 2† to the effect that if a connected and regular point set M is irreducibly connected between two of its points A and B, then M is a simple continuous arc from A to B, is a corollary of Theorem 7 above. For since by Theorem 7 such a set, M, is a continuous curve, A and B are the end points of a simple continuous arc, t, of M. It is clear, then, that $M \equiv t$.

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* A connected set M is said to be *irreducibly connected* about one of its subsets, N, if it has no proper connected subset which contains N. See H. M. Gehman, *Concerning irreducibly connected sets and irreducible continua*, Proceedings of the National Academy of Sciences, vol. 12 (1926), pp. 544–547.

† Loc. cit. I might say here that in establishing the first of those results concerning simple closed curves to which Professor Whyburn kindly calls attention in this connection, I found it necessary to prove as a lemma the definition of arc stated in his Theorem 2. I did not mention this in my abstract (this Bulletin, vol. 32 (1926), p. 123, abstract No. 15) and have not yet published the paper. However, the proof which I developed in that connection is quite different from that given by Professor Whyburn as well as from the proof indicated in the present paper.