

## ON CONNECTED AND REGULAR POINT SETS\*

BY R. L. WILDER

In a recent number of this Bulletin† G. T. Whyburn showed that if  $A$  and  $B$  are any two points of a connected and regular point set  $M$ , and  $K$  denotes the set of all points of  $M$  which separate‡  $A$  and  $B$  in  $M$ , then  $K+A+B$  is a closed and bounded point set. In the present paper I shall show that this theorem is susceptible of quite simple proof, and admits of two obvious generalizations which hold in space of  $n$  dimensions. The methods used in proving these generalizations are also employed to show that if  $N$  is a closed and bounded point set which lies in a connected subset of an open subset,  $F$ , of a connected and regular point set,  $M$ , then  $F$  contains a bounded, connected and regular set which contains  $N$ . These results are then applied to give certain theorems concerning continuous curves.

Use will be made of the notions *region of  $M$*  and *simple chain of regions* as introduced in my paper, *The non-existence of a certain type of regular point set*,§ as well as of Theorem 1 of that paper; these extend readily to  $n$ -dimensional space, if in the definition of *region of  $M$*  "circle" is replaced by "sphere." Furthermore, since the Borel property is employed I make note of the following lemma, the proof of which should be quite obvious.||

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† G. T. Whyburn, *Concerning connected and regular point sets*, this Bulletin, vol. 33 (1927), pp. 685-689.

‡ If  $A$ ,  $B$  and  $X$  are points of a connected set  $M$ , then  $X$  is said to separate  $A$  and  $B$  in  $M$  if  $M-X$  is the sum of two mutually separated point sets which contain  $A$  and  $B$  respectively.

§ This Bulletin, vol. 33 (1927), pp. 439-446. This paper will be referred to hereafter as N. E. R.

|| It will be understood hereafter without explicit statement that the results hold for sets imbedded in euclidean space of  $n$  dimensions.

LEMMA. *If  $M$  is a connected and regular point set, and  $N$  is a closed and bounded subset of  $M$ , and  $G$  is a collection of regions of  $M$  which cover  $N$ , then  $G$  contains a finite subset which covers  $N$ .*

(By virtue of the property of regularity, there exists, concentric with the sphere used in determining a region  $R$  of  $M$ , a sphere  $S$  such that all points of  $M$  interior to  $S$  are also points of  $R$ . When it is said, then, that a collection of regions,  $G$ , covers a subset  $N$  of  $M$ , it is meant that every point of  $N$  is within the sphere  $S$  which corresponds to some region of the collection  $G$ .)

DEFINITION. If  $N$  is a subset of a connected set  $M$  and  $P$  is a point of  $M - N$ , then  $P$  is said to *separate  $N$  in  $M$*  if  $M - P$  is the sum of two mutually separated sets each of which contains at least one point of  $N$ .

THEOREM 1. *If  $N$  is a closed and bounded subset of a connected and regular point set  $M$  and  $K$  denotes the set of all points which separate  $N$  in  $M$ , then  $K + N$  is a closed and bounded point set.*

PROOF. Suppose the set  $K + N$  is not closed. Then there is a point  $P$  which is a limit point of this set and which does not belong to it. As  $N$  is closed,  $P$  is a limit point of  $K$ , but not of  $N$ . If  $x$  is any point of  $M$ , distinct from  $P$  in case  $P$  is a point of  $M$ , there exists a region of  $M$  which covers  $x$  and which neither contains  $P$  nor has  $P$  as a limit point. Let  $G$  denote the collection of all such regions. By the above lemma,  $G$  contains a finite set of regions,  $R_1, R_2, \dots, R_n$ , such that every point of  $N$  is in at least one of these regions. For every  $i$  ( $i = 1, 2, \dots, n$ ), let  $P_i$  be a point of  $N$  in  $R_i$ . For every two points  $P_i$  and  $P_{i+1}$  ( $i = 1, 2, \dots, n - 1$ ) there exists, by Theorem 1 of N. E. R., and by virtue of the fact that  $P$  does not separate these points in  $M$ ,\* a

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\* Use is made in this connection and in the proof of Theorem 2 below, of Lemma 2 of my paper *A characterization of continuous curves by a property of their open subsets*, *Fundamenta Mathematicae*, vol. 11(1928), pp. 127-131.

simple chain of regions of  $G$  from  $P_i$  to  $P_{i+1}$ . The regions  $R_1, R_2, \dots, R_n$  together with the regions which go to make up these simple chains form a finite set,  $g$ , of regions of  $G$ , and if  $H$  denotes the set of all points of  $M$  contained in regions of  $g$ , then  $H$  is a connected and bounded set. As  $P$  is not a limit point of  $H$  there exist in the vicinity of  $P$  points of  $K$  which are not in  $H$ . But clearly such points cannot separate  $N$  in  $M$  since  $N$  is a subset of  $H$ . Thus the supposition that  $K+N$  is not closed leads to a contradiction.

To show that  $K+N$  is bounded, let  $G$  be any collection of regions of  $M$  covering  $N$  and proceed as above to establish the existence of a bounded connected set,  $H$ , which is a subset of  $M$  and contains  $N$ . As every point of  $K$  must lie in  $H$ , it is clear that  $K+N$  is bounded.

DEFINITION. If  $A$  and  $B$  are any two distinct subsets of a connected set  $M$ , and  $X$  is a point of  $M - (A+B)$ , then  $X$  is said to *separate  $A$  from  $B$  in  $M$*  if  $M - X$  is the sum of two mutually separated sets which contain  $A$  and  $B$ , respectively.

THEOREM 2. *If  $A$  and  $B$  are any two distinct subsets of a connected and regular point set  $M$ , such that  $A+B$  is closed and bounded, and  $K$  denotes the set of all points which separate  $A$  from  $B$  in  $M$ , then  $K+A+B$  is a closed and bounded set.*

INDICATION OF PROOF. Select  $P$  and  $G$  as in Theorem 1. As  $P$  does not separate  $A$  from  $B$  in  $M$ , there is a connected subset of  $M - P$  which contains a point  $P_1$  of  $A$  and a point  $P_2$  of  $B$ . Hence from  $G$  can be selected a simple chain from  $P_1$  to  $P_2$ .

The corollarys of Theorems 1 and 2 applied to continuous curves\* are obvious.

As a direct consequence of Theorems 1 and 2 and of the fact that the difference of two closed sets is both an  $F_\sigma$  and a  $G_\delta$ ,† we have the following theorem.

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\* In this paper a continuous curve is considered as a closed, connected and regular point set not necessarily bounded.

† See F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig, 1914, p. 306. An  $F_\sigma$  is the sum of a denumerable set of closed sets and a  $G_\delta$  is the set of points common to a denumerable set of open sets.

**THEOREM 3.** *Let  $M$  be a connected and regular point set. Then if  $N$  is any closed and bounded subset of  $M$ , the set of all points which separate  $N$  in  $M$  is both an  $F_\sigma$  and a  $G_\delta$ ; and if  $A$  and  $B$  are any two distinct subsets of  $M$  such that  $A + B$  is closed and bounded, the set of all points which separate  $A$  from  $B$  in  $M$  is both an  $F_\sigma$  and a  $G_\delta$ .\**

By methods similar to those used in proving Theorem 1 and noting in addition the fact that a region in a connected and regular point set is itself a regular set (Theorem 2 of N. E. R.), we have the following result.

**THEOREM 4.** *If  $K$  is a closed subset of a connected and regular point set  $M$  and  $N$  is a closed and bounded subset of a connected subset of  $M - K$ , then  $N$  lies in a bounded, connected and regular subset of  $M - K$  which has no limit point in  $K$ .*

The analog of Theorem 4 for continuous curves may be stated as follows.

**THEOREM 5.** *If  $N$  is a closed and bounded subset of an open subset,  $Q$ , of a continuous curve  $M$ , and  $N$  lies in some connected subset of  $Q$ , then  $N$  lies in a bounded continuous curve which is a subset of  $Q$ .*

**PROOF.** Hahn has shown<sup>†</sup> that if  $P$  is any point of  $M$  and  $r$  is any positive number, there exists a continuous curve  $M(P, r)$  which is a subset of  $M$ , contains every point of  $M$  less than a certain distance  $d$  (dependent on  $r$ ) from  $P$ , and is such that all of its points are at a distance less than  $r$  from  $P$ . The set of all points  $\{x\}$  of  $M$  such that  $x$  is joined to  $P$  by a connected subset of  $M$  every point of which is at a distance less than  $d$  from  $P$  constitutes a region of  $M$  and

\* In this connection it may be of interest to note that it has been shown that the set of all cut points of a continuous curve is an  $F_\sigma$ . See C. Zarankiewicz, *Sur les points de division dans les ensembles connexes*, *Fundamenta Mathematicae*, vol. 9 (1927), pp. 124–171, Theorem 17.

† H. Hahn, *Mengentheoretische Charakterisierung der stetigen Kurve*, *Wiener Akademie Sitzungsberichte*, vol. 123, Part IIa, pp. 2433–2489; see Theorem XXI, p. 2475. Although Hahn states his result for a bounded regular continuum, it is clear that it holds for any regular continuum.

will be denoted by  $M'(P, r)$ . Clearly  $M'(P, r)$  is a subset of  $M(P, r)$ .

If  $P$  is any point of  $Q$ , let  $r$  be a positive number less than the distance\* from  $P$  to  $M-Q$ . Then the sets  $M(P, r)$  and  $M'(P, r)$  are subsets of  $Q$ . By a method similar to that used in proving Theorem 1 it can be shown that there exists a connected set  $H'$  which consists of a finite number of the regions of type  $M'(P, r)$  and contains  $N$ . The set  $H$  composed of the sets  $M(P, r)$  associated with those sets  $M'(P, r)$  which constitute  $H'$  is a bounded continuous curve lying in  $Q$  and containing all points of  $N$ .

Theorem 5 is a generalization of a result obtained by R. L. Moore† to the effect that if  $Q$  is an open subset of a continuous curve and  $A$  and  $B$  are two points which lie in a connected subset of  $Q$ , then  $A$  and  $B$  are joined by a simple continuous arc which lies wholly in  $Q$ . As I have shown elsewhere‡ that this property is sufficient that a continuum be a continuous curve, it follows that the property stated in Theorem 5 also serves to characterize a continuous curve.

As I have indicated in a recent paper,§ subsets of a point set  $M$  may be separated in  $M$  in different senses. For our present purposes we employ the following definitions.

DEFINITION. If  $N$  is a subset of a connected set  $M$  and  $P$  is a point of  $M-N$ , then  $P$  is said to *separate*  $N$  in  $M$  in the weak sense if there exist two points of  $N$  which do not lie in a connected subset of  $M-P$ .

\* That is, the greatest lower bound of all distances  $Px$ , where  $x$  is a point of  $M-Q$ .

† *Concerning continuous curves in the plane*, *Mathematische Zeitschrift*, vol. 15 (1922), pp. 254-260, Theorem 1.

‡ *Concerning continuous curves*, *Fundamenta Mathematicae*, vol. 7 (1925), pp. 340-377, Theorem 18.

§ *A characterization of continuous curves by a property of their open subsets*, *Fundamenta Mathematicae*, vol. 11 (1928), pp. 127-131. The terminology "separates in the strong (or weak) sense" should not be confused with "disconnects in the strong (or weak) sense" as introduced by R. L. Moore in *Concerning the cut-points of continuous curves and of other closed and connected point-sets*, *Proceedings of the National Academy of Sciences*, vol. 9 (1923), pp. 101-106.

DEFINITION. If  $A$  and  $B$  are any two distinct subsets of a connected set  $M$  and  $X$  is a point of  $M - (A + B)$ , then  $X$  is said to *separate  $A$  from  $B$  in  $M$  in the weak sense* if  $M - X$  contains no connected subset which contains points of both  $A$  and  $B$ .

As examples of cases where these two definitions of "separate" are satisfied, but where the preceding definitions\* are not satisfied, consider the following examples.

*Examples.* Let  $M_n$  ( $n = 1, 2, 3, \dots$ ) denote the straight line interval joining the points  $(0, 0)$  and  $(1, 1/n)$ . Also, denote the points  $(1/2, 0)$ ,  $(1, 0)$  and  $(0, 0)$  by  $A$ ,  $B$ , and  $X$ , respectively, and let  $M = A + B + \sum_{n=1}^{\infty} M_n$ .

The set  $M$  is connected, but the set  $M - X$  contains no connected set containing  $A$  and  $B$ . Hence  $X$  separates  $A$  from  $B$  in the weak sense. However, there is no separation of  $M - X$  into two mutually separated subsets containing  $A$  and  $B$ , respectively, and thus  $X$  does not separate  $A$  from  $B$  in  $M$  in the strong sense, that is, in the sense of the definition which immediately precedes Theorem 2.

If we let  $N$  denote the set of all points with rational coordinates in the interval  $[0, 1]$  of the  $X$  axis, except  $(0, 0)$ , and define  $M_n$  and  $X$  as above, and let  $M$  now denote the set  $N + \sum_{n=1}^{\infty} M_n$ , it is easy to see that  $X$  separates  $N$  in  $M$  in the weak sense, but not in the strong sense, i.e., in the sense of the definition preceding Theorem 1.

On the basis of Lemma 2 of my paper *A characterization of continuous curves by a property of their open subsets*† we have the following extension of Theorems 1–5.

THEOREM 6. (1) *Theorems 1, 2 and 3 still hold true if "separate" be interpreted to mean "separate in the weak sense";* (2) *Theorem 4 still holds true if the words "N is a closed and bounded subset of a connected subset of  $M - K$ " be replaced by "N is a closed and bounded subset of  $M - K$  such that there is no separation of  $M - K$  into two mutually separated sets each*

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\* That is, the definitions preceding Theorems 1 and 2, respectively.

† Loc. cit.

of which contains points of  $N$ ;" and (3) Theorem 5 still holds true if the words " $N$  lies in some connected subset of  $Q$ " be replaced by "there is no separation of  $Q$  into two mutually separated sets each of which contains points of  $N$ ."

In conclusion we may note the following application of part (1) of Theorem 6 to the theory of irreducibly connected sets.\*

**THEOREM 7.** *If the connected and regular point set  $M$  is irreducibly connected about a closed and bounded set  $N$ , then  $M$  is a bounded continuous curve.*

**PROOF.** Let  $P$  be any point of  $M - N$ . Then  $M - P$  contains no connected subset which contains  $N$ , since  $M$  is irreducibly connected about  $N$ . That is,  $M - N$  is the set of points of  $M$  which separate  $N$  in  $M$  in the weak sense, and accordingly by Theorem 6, part (1), the set  $(M - N) + N = M$  is bounded and closed. Hence  $M$  is a bounded continuous curve.

It may be pointed out that Whyburn's Theorem 2† to the effect that if a connected and regular point set  $M$  is irreducibly connected between two of its points  $A$  and  $B$ , then  $M$  is a simple continuous arc from  $A$  to  $B$ , is a corollary of Theorem 7 above. For since by Theorem 7 such a set,  $M$ , is a continuous curve,  $A$  and  $B$  are the end points of a simple continuous arc,  $t$ , of  $M$ . It is clear, then, that  $M \equiv t$ .

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\* A connected set  $M$  is said to be *irreducibly connected* about one of its subsets,  $N$ , if it has no proper connected subset which contains  $N$ . See H. M. Gehman, *Concerning irreducibly connected sets and irreducible continua*, Proceedings of the National Academy of Sciences, vol. 12 (1926), pp. 544-547.

† Loc. cit. I might say here that in establishing the first of those results concerning simple closed curves to which Professor Whyburn kindly calls attention in this connection, I found it necessary to prove as a lemma the definition of arc stated in his Theorem 2. I did not mention this in my abstract (this Bulletin, vol. 32 (1926), p. 123, abstract No. 15) and have not yet published the paper. However, the proof which I developed in that connection is quite different from that given by Professor Whyburn as well as from the proof indicated in the present paper.