LEVI-CIVITA ON TENSOR CALCULUS

The Absolute Differential Calculus (Calculus of Tensors). By Tullio Levi-Civita. Edited by Enrico Persico. Authorized Translation by Miss M. Long. London and Glasgow, Blackie & Son.

When Einstein arrived at his General Relativity Theory he found ready for use a mathematical instrument in the form of the absolute differential calculus (Bateman had already pointed out its applicability in the theory of space-time transformations). This calculus had been worked out by Ricci and was not well known outside of Italy. The only comprehensive exposition was contained then in a paper prepared for the Mathematische Annalen on the invitation of Klein by Ricci and the author of the book under review. In the years following the introduction of the general relativity, the tensor calculus—as the absolute calculus is now often called—has been much perfected, the most notable development being the introduction of the idea of parallel displacement due again to Levi-Civita, who thus among the living contributed most to the subject. In the more than twenty years that passed between the publication of the Annalen paper and the Italian edition of the present book (1923) very many text books appeared. But although many of them introduce parallel displacement, it always appears as a kind of afterthought; a distinctive feature of the present book is a certain unity of exposition achieved by making parallel displacement the basis of the whole edifice.

The fundamental idea stated in a schematic form is this: in Levi-Civita's exposition the absolute differential calculus appears as a modified form of ordinary differential calculus, the modification consisting mainly in that for the second partial derivatives of a function of n variables (and also more complicated expressions of that type) certain other differential expressions of the second order are substituted; the reason why such a modification appears desirable is that the laws according to which the second partial derivatives are changed under a general transformation of the independent variables are quite complicated and much is to be gained by using instead of them other expressions whose laws of transformation are simple. Both the idea of using such expressions and the form in which they appear is suggested by the theory of surfaces; the original feature of the book under review is the following way in which this suggestion is carried out. The variables (two, in the first place) are interpreted as parameters specifying the position of a point on a surface; the first derivatives of a function are then components of the gradient vector tangent to the surface; the second order differential expressions—which are to be used instead of the second derivatives—are obtained as quantities characterizing the change of the gradient vector as we move from point to point on the surface; this involves the comparison of the gradient vector at different points of the surface which necessitates the consideration of displacement of vectors on the surface. Everybody knows how to move vectors on a

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

plane; it is easy to extend the notion to surfaces which are applicable on a plane, that is, developable surfaces; in the case of a general surface, when he wants to displace a sector along a curve Levi-Civita displaces it in the developable surface which is tangent to the given surface along the curve; this is *his parallel displacement.* He finds an analytic expression for this displacement and thus arrives at the required expressions with a simple law of transformation; these expressions are called the covariant derivatives. The author then abandons geometry without abandoning geometrical language and generalises to more than two variables. The difference between using geometrical language—and conceptions—in questions of analysis and doing geometry is not quite clear to the reviewer but the author seems to draw a line between these two things.

The introduction of covariant differentiation as outlined above occupies the center of the original portion of the book—the first 286 pages of the English edition; what precedes is largely an introduction. The reader who is supposed to know or to believe that there exists a solution of a system of ordinary differential equations is introduced in a very neat way into the theory of total and linear partial differential equations. Then tensors are introduced as algebraic systems and the fundamental difficulty mentioned above with second partial derivatives is pointed out. A geometric chapter follows in which parallel displacement is explained and this ends the first part—Introductory Theories. Next comes a formal introduction of covariant differentiation and the remainder of the second part is occupied by the development of the absolute calculus and includes several interesting and original features. We may mention the sections on *geodesic deviation* in which the contents of the author's recent paper in the Mathematische Annalen is reproduced—these sections as well as the discussion of the geometry of the curvature tensor in three-space in the same (seventh) chapter did not appear in the original Italian edition but this fact is not mentioned in the preface.

The third part—Physical Applications—is also original in the English edition: it is devoted to the theory of relativity. Certainly the most noticeable feature in this part is the careful avoidance of electromagnetism which is barely mentioned a few times. This in spite of the fact that as the author himself says in the preface "Electromagnetism, in common with every other physical phenomenon, now comes within the orbit of General Relativity." The reviewer cannot see that this omission can be justified and in his opinion this forms a grave defect in this otherwise wonderful book. Of the many reasons which could be adduced to establish this point it will be enough to mention only two. The electromagnetic tensor and the electromagnetic energy tensor furnish the most natural illustrations and application of the tensor calculus which certainly must be of importance in a treatment intended in the author's own words to exhibit the fundamental principles of Einstein's general relativity theory as an application of the absolute calculus. Then again, the four-dimensional way of viewing things grows naturally from the consideration of the fundamental equations of electrodynamics without making it necessary to introduce into them any modifications, whereas the mechanical equations have to

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

be modified, adapted to the new point of view. It is true that just this fact makes perhaps the theory of relativity more important and fruitful in the application to mechanics; but both esthetically and didactically it is so much more satisfactory to deduce the general principles on the example of electricity and then to apply them to gravitation.

Aside from that hardly any general criticism can be directed against Levi-Civita's exposition of relativity; it is original, complete (with the above exception) and elegant; of the many departures from the usual presentation perhaps Palatini's introduction of centro-symmetric curved space deserves to be particularly commended. A surprising fact is that Levi-Civita does not use in physical applications the notion of parallel displacement which plays such a fundamental role in the theoretical part and which could help to unify the exposition further if it were used in the discussion of the motion of material particles and of the propagation of light.

Throughout the book the author draws freely on the rich material of Italian works which is scarcely used (with few important exceptions) in other text-books. The result is an increased elegance of exposition.

The style is that of theoretical physics. The author comes very close, in the opinion of the reviewer, to striking the happy mean between the Scilla of the epsilon-delta symbolism and the Charibdis of loose reasoning. A pure mathematician would probably prefer a less careful avoidance of the Scilla; it sometimes becomes hard to form a clear idea of what is being neglected and why, but in view of the character of the material treated it is seldom possible to blame the author or to suggest a better course.

G. Y. RAINICH

KENT ON FINANCE

Mathematical Principles of Finance. By Frederick Charles Kent. Second Edition. xiii+177 pp. *Compound Interest and Annuity Tables.* By Frederick C. Kent and Maud E. Kent. viii+214 pp. New York, McGraw-Hill Book Company, 1927. Bound in one volume.

The second edition of Professor Kent's text differs from the first published in 1924 only by the insertion of six pages on interpolation, made possible by the abbreviation of the chapter on logarithms, now entitled "Interpolation and Logarithms," together with a few plate corrections and changes in the remainder of the book. The problem lists are unchanged.

The text appeals to the reviewer as an average book, neither better nor worse than many that have appeared in the same field in the last fifteen years. The conventional topics are covered in the conventional way. Rather more attention is given to the work of the Federal Farm Loan Board than is usual, but there the increase in material is due rather to the description of the administrative processes of the Board than to additional mathematical material.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use