BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

THE OCTOBER MEETING IN NEW YORK

The two hundred sixty-third regular meeting of the American Mathematical Society was held at Columbia University, on Saturday, October 27, 1928, extending through the usual morning and afternoon sessions. The attendance included the following sixty-seven members of the Society:

R. G. Archibald, A. A. Bennett, Benton, B. L. Brown, Carrié, Dadourian, Darkow, Dehn, Demos, Douglas, Dresden, Eisenhart, Feld, Fiske, Fite, Fort, Gehman, Gill, Gronwall, L. S. Hill, Hille, Hofmann, Hollcroft, Huber, Dunham Jackson, M. I. Johnson, R. A. Johnson, Kaplan, Kenny, Kholodovsky, Kline, Koopman, Mark Kormes, Lefschetz, McGiffert, Marden, Meder, Richard Morris, Mullins, C. A. Nelson, K. E. O'Brien, Ore, Pell-Wheeler, Pfeiffer, Pierpont, Post, Raudenbush, Ritt, Rutledge, Schoonmaker, Seely, Serghiesco, Siceloff, Simons, Smail, P. A. Smith, J. M. Thomas, Trjitzinsky, Weida, Weisner, M. E. Wells, D. E. Whitford, Whittemore, Wiener, W. A. Wilson, Margaret M. Young, Zippin.

At the meeting of the Council, the following persons were elected to membership in the Society:

Mr. Andrew Campbell Berry, Harvard University;
Miss Olive Margaret Hughes, Bryn Mawr College;
Mr. William Thomas Reid, University of Texas;
Professor Allen A. Shaw, University of Arizona;
Dr. Warren Jennison Willis, patent attorney, New York City.

The following members of the London Mathematical Society have entered the American Mathematical Society under the reciprocity agreement since the last meeting:

Mr. Francesco Tavani, London;
Professor T. P. Trivedi, Karachi, India.

Associate Secretary Dresden reported the following elections by mail vote of the Council:

To sustaining membership: Members of the Department of Mathematics, University of Wisconsin;

To ordinary membership:

Dr. Frederick Richard Bamforth, University of Chicago;
Professor David Francis Barrow, University of Georgia;
Mr. Leon Battig, Oberlin College;
Dr. Clifford Bell, University of California at Los Angeles;
Professor Arthur H. Blue, Western Union College;
Mr. Arthur Barton Brown, Harvard University;
Professor Helen Calkins, Sweet Briar College;
Mr. Paul T. Copp, Purdue University;
Mr. Paul Cramer, Purdue University;
Professor Louis Antoine Victor De Cleene, St. Norbert College;
Dr. Miltiades Stavros Demos, Harvard University;
Professor Carl Christian Engberg, University of Nebraska;
Mr. William Irvin Foster, Rochester Junior College;
Mr. Harold Sinclair Grant, University of Pennsylvania;
Professor Samuel Oliver Grimm, Lebanon Valley College;
Captain Elmer Ellsworth Hagler, Jr., United States Army;
Professor Marie Mathilda Johnson, Oberlin College;
Dr. Pierce Waddell Ketchum, University of Illinois;
Mr. Edward August Knobelauch, University of Pennsylvania;
Mr. Trueman Lester Koehler, Lehigh University;
Mr. Semen Arsenijevitch Lepeshkin, Brown University;
Professor Anna Marm, Bethany College;
Sister Mary Bertrand (Walton), Marywood College;
Mr. John Ellsworth Merrill, Case School of Applied Science;
Professor Elmer Beneken Mode, Boston University;
Mr. Joseph Kimbark Peterson, Harvard University;
Mr. Thurman Stewart Peterson, Ohio State University;
Miss Mina S. Rees, Hunter College;
Professor Charles Edward Schroeder, Boston College;
Professor Charles Louis Searey, University of Nevada;
Mr. James Singer, Princeton University;
Professor Carlton W. Smith, State Teachers College, Superior, Wis.;
Professor Atherton Hall Sprague, Amherst College;
Mr. Carl Walther Strom, Luther College;
Professor Emory Earl Walden, Lambuth College;
Professor Charles Ernest Weatherburn, Canterbury University College, Christchurch, New Zealand;
Professor Thomas Payne West, University of Idaho, Southern Branch;
Miss Jean Winston, University of Cincinnati;

Nominees of the Penn Mutual Life Insurance Company, Philadelphia: Messrs. J. E. G. Bryant, M. L. Johnson, Adolph F. Schwartz, Lawrence L. Stevens, Clarence E. Willows.

The Council accepted with thanks the invitation of Lehigh University to hold the Eastern Christmas meeting of 1929 in Bethlehem, Pa.

The Council accepted the recommendation of the committee on colloquia that Professor S. Lefschetz be invited to deliver the colloquium lectures at the summer meeting in 1930 at Brown University.

It was voted to approve the use of the name of the Society in the announcements of the American Yearbook; and the President was authorized and requested to appoint, each time for a period of three years, a representative of the Society on the Editorial Board of the Yearbook.

Professor Bennett presided at the morning session, and Professor Jackson at the afternoon session.

Titles and abstracts of the papers read at this session follow below. The papers of Franklin, Garver, Kasner, Stone, Suschkewitsch, Whyburn, and Zippin were read by title. Dr. Arnold was introduced by Associate Secretary Arnold Dresden, Mr. Rashevsky by Professor J. I. Taylor, and Professor Suschkewitsch by Professor H. H. Mitchell.

1. Professor T. R. Hollcroft: *Invariant postulation.*

The invariant postulation of a manifold, simple or multiple, on a variety in i dimensions is the number of invariants among the coefficients of the variety that are necessary and sufficient for the variety to contain a manifold of given nature. The invariant postulation of a certain manifold is obtained readily from the ordinary postulation in which both the nature and position of the manifold are taken into account. Invariant postulation, although a concept not heretofore used except in the case of points in the plane, is of importance mainly because of the geometric relations revealed by it. Some of these relations have been proved before by other methods, and some are new.

2. Dr. H. E. Arnold: *The rational space quintic curve of the second species and its relation to the rational plane quartic curve.*

The rational space quintic curve of the second species, R_3^5 (II), possesses ∞^1 quadrisecant lines, which are given by a pencil of binary quartics. The first part of the present paper concerns itself with the determination of conditions for special quadrisecants, obtained, in general, in terms of the Morley invariants. (See R. M. Winger, American Journal, vol. 36.) By means of these conditions a correspondence is set up between certain types of R_3^5 (II) and those of R_2^4 (the rational plane quartic curve). The R_3^5 (II) for which I_2 vanishes is determined by a binary quintic form whose roots represent coplanar points. In the second part of the paper several facts concerning this plane are obtained. Finally, some properties of R_3^5 (II) are derived from the Jonquière quartic, and a geometric meaning is given to the unique sextic apolar to the pencil of line sections of R_2^4 which pass through a fixed point of the plane.

3. Dr. Louis Weisner: *Invariants of a plane 5-point.*

From the complete system of invariants of a binary quintic, consisting of I_4, I_8, I_{12}, I_{18}, expressed in terms of the roots, invariants J_6, J_{12}, J_{18}, A_{27} of the associated plane 5-point are found by the method of complementary determinants explained by Coble. But these four invariants of the plane 5-point do not form a complete system because A_{27} turns out to be an alternating function. In the present paper a complete system is found to consist of J_6, J_{12}, J_{18}, J_{33}, the subscripts denoting the degrees of the invariants. The results are applied to a redetermination of the invariants of E. H. Moore's cross-ratio group of order 120.

4. Professor R. G. Archibald: *An arithmetical function analogous to Euler's ϕ-function.*

An arithmetical function $\omega(n)$ is defined as the number of integers relatively prime to n in the sequence $1\cdot 2\cdot 3/6$, $2\cdot 3\cdot 4/6, \cdots, n(n+1)\cdot(n+2)/6$. E. Lucas (*Théorie des Nombres*, vol. I, 1891, p. 403) and R. D. Carmichael (*The Theory of Numbers*, 1914, p. 36) suggested the problem of finding the number of terms in the foregoing sequence which are relatively prime to n. As far as the author knows, a formula giving the number of such terms has not yet been published. In the present paper, by the use of Lucas' ψ-function, an explicit formula for $\omega(n)$ is obtained in terms of n if, where m is an integer relatively prime to 6, n is different from $2^k 3m (k\geqq 0)$. In these cases, solutions of a set of linear congruences (always solvable) are to be obtained. If $\omega_5(30\ m)$ denotes the number of terms in the given sequence relatively prime to 6 m when $n=30\ m$, and if m is any positive integer relatively prime to 6, $\omega(36\ m)=2\omega(18\ m)=\omega(6\ m)+\omega_5(30\ m)=\omega(12\ m)+\omega(24\ m)$. Moreover, if m_1 and m_2 are two relatively prime numbers each relatively prime to 6, and if $k\geqq 2$ and $l\geqq 1$, $\omega(m_1 m_2)=\omega(m_1)\omega(m_2)$, $\omega(2^k m)=\omega(2^k)\omega(m)$, $\omega(3^k m)=\omega(3^k)\omega(m)$, $\omega(2^l 3^k m)=\omega(2^l 3^k)\omega(m)=\omega(2^l)\omega(3^k m)=\omega(3^k)\omega(2^l m)$.

5. Dr. Jesse Douglas (National Research Fellow): *On the inverse problem of the calculus of variations.*

The problem is as follows: given a system of paths, under what conditions can it be identified with the totality of extremals of a calculus of variations problem? We represent the system of paths by the differential equations $d^2x^i/dt^2=\overset{i}{H}_2(x, p)$, where $p^i=dx^i/dt$ (Annals of Mathematics, vol. 29 (1928), pp. 143–168), and the calculus of variations problem by $\int F(x, p)dt=$ minimum. $\overset{i}{H}_2$ is homogeneous of the second degree and F of the first degree in the variables p. Necessary and sufficient conditions are found to be represented by the system of covariant partial differential equations (1) $F_{|i,j}-F_{|j,i}=0$, where the stroke denotes $\partial/\partial p$ and the comma covariant differentiation based on the affine connection $\Gamma^i_{j,k}=\frac{1}{2}\partial^2 \overset{i}{H}_2/\partial p^j\partial p^k$. There are three cases as to (1): (a) it may have no solutions F (the general case); (b) it may have a class of solutions depending linearly on a finite number of arbitrary constants; (c) it may have a class of solutions involving arbitrary functions. In the present paper, sufficient conditions are established for the case (b). The only operations required for the application of these conditions are (covariant) differentiation and the algebraic discussion of a certain finite system of linear equations.

6. Dr. Jesse Douglas: *A generalization of homogeneous functions and of Euler's equations.*

The results here are a by-product of the writer's work on spaces of K-spreads (see this Bulletin, Jan.-Feb., 1928). Let $H^{\lambda\mu\cdots\nu}_{\alpha\beta\cdots\gamma}$, where there are r lower and s upper indices each varying from 1 to K, be a system of functions of the variables p^i_α; $i=1, 2, \cdots, N$; $\alpha=1, 2, \cdots, K$; we imagine the system p^i_α to be distributed into N sets of K variables, with i fixed for each set. Let these sets undergo cogredient linear transformation: $q^i_\alpha = A^\beta_\alpha p^i_\beta$ reciprocally $p^i_\alpha = B^\beta_\alpha q^i_\beta$, the coefficients A^β_α, B^β_α being therefore the same for every value of i. Suppose that, identically,

$$H^{\lambda\mu\cdots\nu}_{\alpha\beta\cdots\gamma}(q) = A^\rho_\alpha A^\sigma_\beta \cdots A^\tau_\gamma B^\lambda_\phi B^\mu_\psi \cdots B^\nu_\omega H^{\phi\psi\cdots\omega}_{\lambda\mu\cdots\nu}(p). \tag{1}$$

The property expressed by this equation reduces to homogeneity of degree $r-s$ in case $K=1$. Generalizations of Euler's equations of the first and higher orders are established. Generalization is further made to the case where (1) involves as factor in its second member a power of the determinant $|A^\beta_\alpha|$ (relative tensor); also to the case where the independent variables form N systems ${}^i p^{\lambda\mu\cdots\nu}_{\alpha\beta\cdots\gamma}$, which undergo cogredient tensor transformation.

7. Mr. Nicholas Rashevsky: *An electrostatic problem.*

The solution of the electrostatic problem for a space limited by two infinite parallel planes, one of which is provided with a set of equidistant parallel blades, at right angles to the planes, is given in the form of an infinite product. The use of the first factor of the product as the first approximation leads to elliptic integrals. By a logarithmic transformation, the above solution gives also the solution of the corresponding problem for two coaxial cylinders, one of which is provided with equidistant radial blades. The solution is applied to the calculation of voltage factors in certain special types of three-electrode tubes.

8. Dr. Lulu Hofmann: *Remarks on a certain aspect of plane projective transformations.*

In two projectively related planes, the metric properties of corresponding segments on corresponding lines and corresponding angles at corresponding points are known. This paper investigates the metric properties of corresponding segments radiating out from corresponding points. The principal result is as follows. With every fixed pair of corresponding points P_0, P_0' in two projective planes π, π', and every value of a positive quantity k, is associated a pair of elliptic quartics Q_k, Q_k', corresponding under the projectivity, which are the loci of all pairs of corresponding points P, P', such that the distances of any two such points from the respective fixed points have the same ratio: $\overline{PP_0}=k\overline{P'P_0'}$. These quartics $Q_k[Q_k']$ have one double point at the infinite point of the vanishing line of $\pi[\pi']$ and the other at the fixed points $P_0[P_0']$. They consist of two odd branches lying on different sides of the respective vanishing lines and having the equi-segmental axes for asymptotes. They are tangent to the circles about $P_0[P_0']$ as centers at the cyclic points. In particular for $k=1$, $Q_1[Q_1']$ represents in $\pi[\pi']$ the locus of the double points determined by the projectivity as $\pi'[\pi]$ is successively laid on $\pi[\pi']$ in the ∞^1 ways possible such that P_0 and P_0' coincide.

9. Professor Edward Kasner: *Polygenic functions of two complex variables.*

In a paper read at the Bologna Congress, the author began the study of polygenic functions of two variables by showing that the partial derivatives are represented by a pair of uniformly parametrized circles or *clocks*. (For the case of one variable, see Science, Dec. 16, 1927.) He now studies the three partial derivatives of second order, which give rise to quite complicated configurations of interest. (For the case of one variable, see the Transactions of this Society, Oct., 1928.)

10. Professor Raymond Garver: *A solution of the quartic equation.*

By the use of two theorems in the author's paper recently published in the Messenger of Mathematics, the quartic equation is solved in a manner that seems to present some interest. The present paper will appear in the American Mathematical Monthly.

11. Professor M. H. Stone: *Hermitian symmetric operators in abstract Hilbert space.*

The present paper resolves the characteristic value problem for an operator T with the following properties: (a) T orders to each element of a set $\mathfrak{F}$ everywhere dense in an abstract complex Hilbert space $\mathfrak{H}$ an element of $\mathfrak{H}$; (b) if f and g are elements of $\mathfrak{F}$, $Q(f, Tg)=Q(Tf, g)$, where Q is a Hermitian bilinear form determining the metric of $\mathfrak{H}$; (c) if two elements f and f^* exist in $\mathfrak{H}$ such that, for every g in $\mathfrak{F}$, $Q(f, Tg)=Q(f^*, g)$, then f belongs to $\mathfrak{F}$. The theory includes the known theory of operators for which $\mathfrak{F}\equiv\mathfrak{H}$, developed by Hilbert and others, and leads to analogous results. It includes also the results obtained by J. von Neumann for "real" operators, as stated by him in the Göttinger Nachrichten, 1927, Heft 1, pp. 1–57.

12. Mr. P. M. Swingle: *End sets of bounded continua irreducible between two points.*

A study is made of continua, irreducible between two points, based particularly upon various types of their subsets called "end sets." Among other results, necessary and sufficient conditions for the indecomposability of such continua are obtained.

13. Professor Philip Franklin: *The operators of quantum mechanics.*

In this note we study some properties of the transition from the Hamiltonian function to the Schrödinger operator. In particular we show that any attempt to define the correspondence by linear operators must fail to yield a result independent of coordinates.

14. Professor Philip Franklin: *Algebraic matric equations.*

The equations here studied are of the form

$$P(X) = A,$$

where the left member is a polynomial, and A and X are matrices. For the special case

$$X^n = A,$$

we determine a general form for X giving all the roots. With some restrictions on X, the results are extended to the more general case. In particular, after further specializing X to be expressible as a polynomial in A, we obtain from our results those of Roth (Transactions of this Society, vol. 30, p. 579).

15. Professor Anton Suschkewitsch: *On a generalization of the associative law.*

In this paper it is pointed out that the theorem of Lagrange to the effect that the order of a group is divisible by the order of a subgroup may be proved without the complete use of the associative law. A generalization of this law is proposed, and some theorems regarding it are established.

16. Professor G. T. Whyburn: *Local separating points of continua.*

The point P of a continuum M will be called a *local separating point* of M if there exists a compact open set R containing P and such that if C is the component of $M \cdot \overline{R}$ which contains P, then $M \cdot \overline{R} - P$ is separated between some two points of $C - P$. The point P of an E-set M is called an *indispensable point* of M if $M - P$ is not an E-set. In this paper a study is made of the local separating points of continua M lying in any locally compact, metric, and separable space. These points are characterized in various ways, some in terms of the indispensable points of certain cuttings of M, and theorems are obtained elucidating their properties. Perhaps the most interesting ones are the following: (1) all save possibly a countable number of the local separating points of M are points of order two of M; (2) every uncountable set of local separating points of M contains an uncountable subset every pair of points of which cuts M. The notion of a *local separating continuum* is introduced, and many important properties of local separating points are shown to hold for local separating continua.

17. Professor Edward Kasner: *Higher partial derivatives of polygenic functions.*

The higher partial derivatives of polygenic functions depend upon the curves of approach and also upon the correspondence between the curves. The order of differentiation must be taken into account. Thus in general

$$\frac{\partial^2 w}{\partial z_1 \partial z_2} \neq \frac{\partial^2 w}{\partial z_2 \partial z_1}$$

In fact equality, for all approaches, holds only when w is an analytic function. Thus there are in general four distinct derivatives of second order. For rectilinear approach these reduce to three. Higher orders are also discussed.

18. Professor Norbert Wiener: *The summability of Fourier series.*

The author applies his new method in Tauberian theorems to the demonstration of the Hardy-Littlewood theorems on the convergence and summability of Fourier series.

19. Mr. Leo Zippin: *The Janiszewski-Mullikin theorem.*

The author proves that in a one-dimensional continuous curve containing at least one simple closed curve the Janiszewski-Mullikin theorem is not satisfied. In an acyclic continuous curve it is vacuously satisfied. For the proof, he requires a theorem of Gehman's on subsets of a plane continuous curve which lie on an acyclic subcontinuous curve which he generalizes to n dimensions.

20. Professor W. A. Wilson: *Certain problems related to the cutting of a simply connected plane region by a continuum.*

Let R be a simply connected plane region with a bounded frontier F and C be a bounded continuum contained in $\overline{R}$ such that $C \cdot F$ is the sum of two closed sets α and β between which C is irreducible and C is neither indecomposable nor the union of two indecomposable continua. A study is made of the frontiers of the regions into which C divides R for various types of frontiers F. When C is a simple arc and F is a regular frontier, the nature of the frontiers depends upon whether F contains 0, 1, or 2 continua irreducible between α and β, and the various possibilities are catalogued. If C is an arc, F is the irreducible sum of two continua H and K neither of which disconnects F, and the ends of C lie on different components of $H \cdot K$, the frontiers are $C+H$ and $C+K$. Various sufficient conditions for the extension of this last result to the case that C is not a simple arc are given. In this connection it is also shown that, if F is the union of two continua H and K and is disconnected by neither of them precisely two components of $H \cdot K$ are accessible by a continuum from R and can therefore be joined by an irreducible continuum contained in $\overline{R}$ and connected im kleinen at all of its points save possibly those on $H \cdot K$.

21. Professor W. A. Wilson: *Some properties of multicoherent continua.*

This paper appears in full in the present issue of this Bulletin.

22. Professor J. F. Ritt: *On the zeros of exponential polynomials.*

By an exponential polynomial is meant a function $a_0e^{\alpha_0 x}+\cdots+a_ne^{\alpha_n x}$, with constant a's and α's. The distribution of the zeros of such functions has been studied by Tamarkin and by Pólya. In the present paper it is proved that if the quotient of two exponential polynomials is an integral function, then the quotient is an exponential polynomial.

23. Professor B. O. Koopman: *On the singularities of linear partial differential equations.*

We consider the system of m equations of any order in m unknown functions, in the independent variables $(x, \cdots, x_n)$, and having analytic coefficients. A slight modification of the Cauchy-Kovalevsky calculus of limits establishes the following: If R_{2n} is a $2n$-dimensional region of the complex $x_1 \cdots x_n$-space containing no singular point of the system, and S_{2n-2} a regular portion of the analytic surface

$$S(x_1, \cdots, x_n)=0,$$

in R_{2n}, not tangent to a characteristic, there exists a region P_{2n} of R_{2n}, containing S_{2n-2}, throughout which every solution of the Cauchy problem is analytic, whatever be the (analytic) Cauchy data on S_{2n-2}. From this it follows that if

$$f(x_1, \cdots, x_n) = 0$$

is an isolated analytic singularity of an analytic solution of the equations, and if it contains only ordinary points, it is a characteristic. These results have been obtained in the case of two independent variables by LeRoux and Delassus by methods which do not admit of extension; and by Delassus and Hadamard for n variables under very restrictive assumptions.

24 . Dr. Morris Marden (National Research Fellow): *Zero-free regions of a certain partial fractions.*

This paper tries to find simple regions of the plane which do not contain any zeros of the partial fraction

$$\sum_{i=1}^{n}\alpha_i/(z-a_i),$$

when α_i is a complex constant. It presents a few such regions where the points a_i are collinear, or where the numbers a_i are all real or appear in conjugate imaginary pairs, as well as in cases where the points a_i are arbitrarily distributed in the plane.

25. Dr. T. C. Benton: *On continuous curves which are homogeneous except for a finite number of points.*

A continuous curve is said to be homogeneous if for every pair of points x and y belonging to the curve there exists a (1, 1) continuous correspondence of the *curve* into itself which makes x correspond to y. This paper classifies the plane continuous curves which are homogeneous except for a finite number of points. For one non-homogeneous point c, the set consists of a finite ($\neq 2$) or countably infinite set of simple closed curves which have only c in common. For two such points, the set consists of a finite number ($\neq 2$) of arcs joining the two points. For three or more points a skeleton set is formed by replacing by a single arc the set of arcs which join two non-homogeneous points but contain no other non-homogeneous points. The skeleton set may be either a set of arcs having only one point in common or a simple closed curve, or a set which is in (1, 1) correspondence with the projection of one of the regular polyhedra or of one of the semi-regular polyhedra on one of its faces in such a way that none of the projections of edges have any intersections except projections of vertices. The case of unbounded plane homogeneous continuous curves is also taken up and it is found that except for the one-point case where the set corresponds to a finite number of rays leaving the non-homogeneous point, there is no such set other than the whole plane.

ARNOLD DRESDEN,
Associate Secretary