GRASSMANN'S PROJECTIVE GEOMETRY, VOLUME II

Projektive Geometrie der Ebene unter Benutzung der Punktrechnung. Volume II: *Ternäres.* Part 1. By Hermann Grassmann. Leipzig und Berlin, B. G. Teubner, 1913. xii+410 pp.

Projektive Geometrie der Ebene unter Benutzung der Punktrechnung. Volume II: *Ternäres.* Part 2. By Hermann Grassmann. Leipzig und Berlin. B. G. Teubner, 1927. xvi+522 pp.

The first volume of this work was published in 1909 and reviewed in this Bulletin (1913) by L. W. Dowling. The first part of the second volume appeared in 1913, but, owing to the world war, the volume was not completed before December 1921, when the finished second part was deposited as manuscript in the author's desk. Hermann Grassmann, Jr. died the following month, Jan. 21, 1922. One of his students, G. Wolff in Hannover, took charge of the publication of this second part, but owing to financial difficulties the work did not appear before 1927. Among those giving financial aid we note our own E. Carus.

The first part, which has four chapters (Hauptteile), introduces us to the ternary field of projective geometry. A point x in the plane is given by the equation $x=\xi_1e_1+\xi_2e_2+\xi_3e_3$, where e_1, e_2, e_3 are the vertices of the fundamental triangle; ξ_1, ξ_2, ξ_3, the "Ableitszahlen," may be considered as the Grassmann coordinates of the point x. The dual of the point is the "stab," or directed line-segment which is bound to the line on which it lies, in the sense that it can only be carried along the line in its own direction or in the opposite direction. Any stab U in the plane is represented by the equation $U=u_1E_1+u_2E_2+u_3E_3$, where u_1, u_2, u_3 are the Grassmann line-coordinates and E_1, E_2, E_3 are "stabe" on the lines joining the fundamental points, or "grundstäbe," defined by the equations $E_1=[e_2e_3]$, $E_2=[e_3e_1]$, $E_3=[e_1e_2]$. The unit point e, mass m, is now defined as a point with coordinates (1, 1, 1), so that $e=e_1+e_2+e_3$; the masses m_1, m_2, m_3 may be so chosen that the exterior three-point product $[e_1e_2e_3]=1$ (blatteinheit). Dually we have a unit "stab" of length l such that $E=E_1+E_2+E_3$ and $[E_1E_2E_3]=[e_1e_2e_3]=1$. It is then shown that the ratios $\xi_1:\xi_2:\xi_3$ and $u_1:u_2:u_3$ may be geometrically interpreted as double ratios

$$\frac{\xi_2}{\xi_3}=\frac{p_2}{p_2'}:\frac{p_3}{p_3'},\qquad \frac{\xi_3}{\xi_1}=\frac{p_3}{p_3'}:\frac{p_1}{p_1'},\qquad \frac{\xi_1}{\xi_2}=\frac{p_1}{p_1'}:\frac{p_2}{p_2'},$$

where p_1, p_2, p_3 are the distances of the point x from the sides of the fundamental triangle, and p_1', p_2', p_3' those of the unit point e from the same sides respectively. Dually we obtain for the ratios $u_1:u_2:u_3$ the values

$$\frac{u_2}{u_3}=\frac{q_2}{q_2'}:\frac{q_3}{q_3'},\qquad \frac{u_3}{u_1}=\frac{q_3}{q_3'}:\frac{q_1}{q_1'},\qquad \frac{u_1}{u_2}=\frac{q_1}{q_1'}:\frac{q_2}{q_2'},$$

q_1, q_2, q_3 being the lengths of the perpendiculars from the vertices e_1, e_2, e_3 on the stab U, and q_1', q_2', q_3' those of the perpendiculars from the same three vertices on the unit stab E. By letting E_3 recede to infinity the author obtains the Hessian coordinate system familiar to students of Clebsch-Lindemann. The chapter concludes with a section on the harmonic relations in the quadrilateral-quadrangular configuration.

The Grassmann algorithm when applied to projective geometry does not however produce a pure coordinate geometry. The coordinates ξ_i and u_i are used in proving certain important formulas involving exterior and combinatory products, and the geometric and projective properties are then derived from the interpretation of these formulas; in other words, direct analysis is employed as far as possible. That coordinates cannot be entirely dispensed with seems to substantiate Study's dictum, at least in this case, that "without coordinates there is no geometry."

The second chapter deals with the fundamental properties of collineations, their double-elements and certain combinatory products of two and three collineations. A collineation k is defined by the symbolic or "extensive" fraction

$$k = \frac{a_1, a_2, a_3}{e_1, e_2, e_3},$$

which means that $e_1k = a_1$, $e_2k = a_2$, $e_3k = a_3$, and that $(\xi_1e_1 + \xi_2e_2 + \xi_3e_3)k = \xi_1a_1 + \xi_2a_2 + \xi_3a_3$, or $xk = y$; to the vertices e_1, e_2, e_3 of the fundamental triangle correspond the vertices a_1, a_2, a_3 of another triangle. Any point-row $y + hz$ is transformed into a point-row $yk + hkz = (\xi_1 + h\zeta_1)a_1 + (\xi_2 + h\zeta_2)a_2 + (\xi_3 + h\zeta_3)a_3$, and the double-ratio of four points x, y, z, u on a line is invariant.

Closely related to the collineation k is the so-called adjoint collineation

$$K = \frac{A_1, A_2, A_3}{E_1, E_2, E_3}$$

which is a line-to-line transformation (Stab-Stab Abbildung) carrying a stab U into another stab UK, that is $U = \sum u_iE_i$ is transformed into $V = \sum u_iA_i$, where $E_1 = [e_2e_3]$, $E_2 = [e_3e_1]$, $E_3 = [e_1e_2]$, $A_1 = [a_2a_3]$, $A_2 = [a_3a_1]$, $A_3 = [a_1a_2]$. From the equation $[x, U] = 0$, which means that x is on U, it follows that $[xk, UK] = 0$, that is, the transform xk of the point x by k is on the transform UK of U by K.

If k and l are two collineations, the combinatory product $[kl]$ is defined as follows:

$$(1) \qquad [kl] = \frac{[yz \cdot kl]}{[yz]} = \frac{[yk \cdot zl] - [zk \cdot yl]}{2[yz]},$$

the right side of which is shown not to depend on the two points y and z; hence the symbol $[kl]$ is justified. If then in (1) we put $y = e_1$, $z = e_2$; $y = e_2$, $z = e_3$; $y = e_3$, $z = e_1$ in succession, it is easily seen that $[kl]$ may be put in the form of an extensive fraction

$$[kl] = \frac{[e_2e_3 \cdot kl], [e_3e_1 \cdot kl], [e_1e_2 \cdot kl]}{[e_2e_3], [e_3e_1], [e_1e_2]}.$$

If $k = l$, we have $[k^2] = K$, an important formula.

The combinatory product of three collineations is defined thus:

$$(2)\quad [klm] = \frac{[xyz\cdot klm]}{[xyz]} = \frac{\begin{array}{c}[xk\cdot yl\cdot zm] + [yk\cdot zl\cdot xm] + [zk\cdot xl\cdot ym]\\ -[xk\cdot zl\cdot ym] - [yk\cdot xm\cdot zl] - [zk\cdot ym\cdot xl]\end{array}}{6[xyz]},$$

and it is shown that the fraction on the right side does not depend on the points x, y, z, only on the numerators a_i, l_i, c_i of the extended fractions k, l and m. If $k=l=m$, we have the combinatory cube

$$[k^3] = \frac{[a_1a_2a_3]}{[e_1e_2e_3]} = \frac{[e_1k\cdot e_2k\cdot e_3k]}{[e_1e_2e_3]}$$

which is another fundamental formula. The author then proceeds to classify the various singular collineations, namely those for which $[k^3]=0$:

1. $[a_1a_2a_3]=0$, at least one of the products $[a_ia_k]$ being different from zero.
2. $[a_1a_2a_3]=0$, all the products $[a_ia_k]$ vanishing.
3. $[a_1a_2a_3]=0$, all three a's vanishing.

Starting with the adjoint K and the adjoint to K, denoted by $\bar{k}$,

$$\bar{k} = \frac{[A_2A_3],[A_3A_1],[A_1A_2]}{[E_2E_3],[E_3E_1],[E_1E_2]}$$

and introducing the combinatory product $[Kl]$ of two stab-stab collineations, he proves that $[K^2]=\bar{k}=ak$, where $a=[a_1a_2a_3]=[k^3]$. The combinatory product $[KLM]$ is then defined in a manner analogous to that for $[klm]$ and an expression for $[K^3]$ is obtained, viz.: $[K^3]=[A_1A_2A_3]$ $=[a_1a_2a_3]=a$. A classification of singular stab-stab collineations then follows, dual to the above for point-collineations.

It appears here, as elsewhere throughout the work, that the author discusses the dual case in detail instead of merely stating the result, applying the principle of duality, as is usually done, in projective geometry. This tends to increase materially the number of calculations and becomes wearisome in the long run, but the elementary student who attacks the subject for the first time *via* Grassmann will perhaps be satisfied, as he is saved the trouble of proving the dual case. It is all "cut and dry." In the second part of the volume he sometimes, but not often, omits the calculation.

The double elements of the collineation are obtained starting with the condition $d_tk=r_td_t$ where d_t is a point and r_t a pure number. The question is then: What point, or points d_t are left in situ in a collineation k, the point d_t changing at most its mass which is multiplied by a number r_t? The answer to this question leads to a cubic equation having r_1, r_2, r_3 for roots. This is *the characteristic equation*. We are thus led to a collineation

$$k = \frac{r_1d_1, r_2d_2, r_3d_3}{d_1, d_2, d_3},$$

d_1, d_2, d_3 being the fundamental triangle. The dual case, which is carried out in detail, gives rise to the stab-stab collineation

$$K = \frac{r_2r_3[d_2d_3], r_3r_1[d_3d_1], r_1r_2[d_1d_2]}{[d_2d_3], [d_3d_1], [d_1d_2]}.$$

The various subcases are then considered, the result being a slightly modified form of the classification found in Veblen and Young's treatise, vol. I, pp. 271–276. It may be noted in this connection that Grassmann needs 25 pages to exhaust the subject of double elements and types of collineations, while Veblen and Young need only 5 pages. This is not due to a more complicated algorithm, for the calculations in Grassmann are simple, and the geometry can be read from the symbolic equations without difficulty, but rather to a more detailed consideration of the geometry involved. In this respect the method of Grassmann is carried out very carefully; he never forgets that he is a geometer.

The last section (29) of this chapter deals with the geometric meaning of the vanishing of the combinatory product $[klm]$ which was defined above, (eq. 2). If $l=m$ we get the product $[kll]=0$ or

$$[xkyz] + [ykzx] + [zkxy] = 0. \tag{3}$$

If (3) is satisfied the collineation has an *inscribed triangular position.* This theorem is due to Pasch. A few more theorems of a similar kind then follow. The remainder of the chapter deals with point-triples. The locus of points x such that the two pencils $xabc$ and $xa'b'c'$ are in involution is a cubic called the cubic of involution; the special case when this cubic degenerates into three lines, i.e. when a, b, c and a', b', c' are collinear point triples is then considered, and a number of theorems on point-triples are deduced.

In the sixth chapter the author considers first correlations in the plane and then the more special polarities with their pole-curves of the second order and class. He sets up the extensive fractions for correlations and and polarities together with a number of combinatory products of two and three correlations (polarities). Starting with these formulas, the remaining part of the chapter is devoted to the consideration of poles and polars, the polar triangle, curves (that is pole-curves) of the second order and class, etc. The discussion of singular or "entartete" systems of polarity occupies about 30 pages, and the various forms of conics, considered as point-loci and stab-loci and depending on the various positions of their polar triangles, another 57 pages. The last section deals with conics in Cartesian point-coordinates (oblique axes) and Hessian line-coordinates, one side of the polar triangle being at infinity. The last, but not the least interesting, chapter deals with the theory of pencils of conics and their duals, ranges of conics. We shall not give an account of the various cases considered. The work is done with meticulous care, and the dual case (ranges) is treated in detail. The author finds no place for a discussion of group-properties of collineations, correlations and polarities, although in the first volume he did consider the product (folgeprodukt) kl of two projectivities in the binary field. The notion of an invariant, as well as invariant configurations connected with the various subgroups of projective transformations, is also absent. Nevertheless, the first part of this volume is a decided success from the standpoint of the Grassmann point-calculus; he no doubt had two main objects in view: To write a projective geometry based on the Grassmann point-calculus, and to make it a magisterial work that a student can easily master. Any mathematician who has found the elder Grassmann's presentation of his point-calculus, such as is found in Grassmann's *Ausdehnungslehre* of 1862 rather "abschreckend" will be pleasingly surprised at the ease with which the calculus can be mastered when presented in a masterly way.

The second part begins with a section on the projective geometry on a curve of the second order and class. Involutions on a conic are carefully considered and a following short section deals with the equianharmonic projectivities.

In the next chapter the author turns to the consideration of range-pencils and pencil-ranges of conics. Special cases are considered such as homo-asymptotic hyperbolas and ellipses. All collineations are then found that leave pencil-ranges invariant, and incidentally also the collineations which transform a conic into itself. The general theory of reciprocities (correlations) is treated in the remainder of the chapter. Let

$$r = \frac{A_1, A_2, A_3}{e_1, e_2, e_3}, \quad R = r^2 = \frac{a_1, a_2, a_3}{E_1, E_2, E_3}$$

be a reciprocity and its adjoint. If $[A_1A_2A_3] \neq 0$, and $[a_1a_2a_3] \neq 0$, we have also the inverse

$$\frac{1}{r} = \frac{e_1, e_2, e_3}{A_1, A_2, A_3}, \qquad \frac{1}{R} = \frac{E_1, E_2, E_3}{a_1, a_2, a_3};$$

for the special case of polarity we have $r = a/R$, $a = [a_1a_2a_3]$. The curves $[x \cdot xr] = 0$, and $[U \cdot UR] = 0$ are called the *nuclei* of the reciprocity. The first expresses the condition that a point shall lie on the corresponding line, and the second equation that the line U shall pass through the corresponding point. These curves are also called respectively *the pole-curve* and the *polar curve*. The reciprocity conjugate to r is

$$r' = \frac{A_1', A_2', A_3'}{e_1, e_2, e_3} = \frac{a}{R},$$

where $A_i' = a_{1i}E_1 + a_{2i}E_2 + a_{3i}E_3$. A null-system of the second order is one for which $a_{ii} = 0$, $a_{ik} = -a_{ki}$, and for a null-system of the second class we have $A_{ii} = 0$, $A_{ik} = -A_{ki}$, the A_{ik} being the minors of the determinant $|a_{11}a_{22}a_{33}|$. A reciprocity of this kind may be put in the form

$$n = \frac{[ae_1], [ae_2], [ae_3]}{e_1, e_2, e_3},$$

in which a stands for the null-point of the reciprocity. Dually we get

$$N = \frac{[AE_1], [AE_2], [AE_3]}{E_1, E_2, E_3},$$

where A is the null-axis of the reciprocity. Of these two singular correlations, the one carries a point x into a line nx joining x to the null-point a so that the pole-curve becomes a double point a, and the second carries a line U into the point of intersection of the line with the null-axis, i.e. A, as a double line, is the polar curve.

The author then discusses conjugate and adjoint reciprocities and the relative positions of their nuclei. Although for a reciprocity, which is not a polarity, we must always have $r \neq r'$, the equality $cr = cr'$ may be satisfied for a special point. This point is shown to have the coordinates $c_1 : c_2 : c_3 = a_{23} - a_{32} : a_{31} - a_{13} : a_{12} - a_{21}$ and is called the "kernpunkt" of the reciprocity, and dually, for the coordinates of the "kerngerade" $C_1 : C_2 : C_3 = A_{23} - A_{31} : A_{32} - A_{13} : A_{12} - A_{21}$, and he proves that the "kernpunkt" of a reciprocity is the pole of the "kerngerade" with respect to both nuclei of the reciprocity.

In the tenth and eleventh chapters the author takes up the subject of apolarity. He derives the condition for apolarity of two reciprocities in the form

$$(4) \qquad [rS] = \frac{1}{3[xyz]}\{[xr \cdot XS] + [yr \cdot YS] + [zr \cdot ZS]\} = 0,$$

where

$$X = [yz], \quad y = [zx], \quad Z = [xy], \quad S = [st].$$

If in this equation we put $e_i r = A_i$ and $E_i S = b_i$, $x = e_1$, $y = e_2$, $z = e_3$, $X = E_1$, $Y = E_2$, $Z = E_3$, the condition $[rS] = 0$ takes the well known form

$$a_{11}B_{11} + \cdots + (a_{23}B_{23} + a_{32}B_{32}) + \cdots = 0,$$

and when r and S are two polar systems p and Q, $a_{11}B_{11} + \cdots + 2a_{23}B_{23} + \cdots = 0$.

In Chapter XI the author introduces two new symbolic expressions, the *Lückenform* and the *Potenzform*. We shall not try to give English equivalents for these terms; *lacunary forms* would perhaps do for the first term, but *powerform* does not sound well, and we shall therefore use the German terms throughout. Suppose we have a polarity of the second order

$$p = \frac{A_1, A_2, A_3}{e_1, e_2, e_3},$$

where $A_i = \sum a_{ik}E_k$ and $a_{ik} = a_{ki}$; then if $x = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3$, we have the polar $xp = \xi_1 A_1 + \xi_2 A_2 + \xi_3 A_3$. Multiplying x exteriorly by E_1, E_2, E_3 we get $[xE_1] = \xi_1$, $[xE_2] = \xi_2$, $[xE_3] = \xi_3$ and the polar xp becomes

$$xp = [xE_1]A_1 + [xE_2]A_2 + [xE_3]A_3 = \sum [xE_i]A_i,$$

and the quadratic form $[x \cdot xp]$ is represented by the expression

$$[x \cdot xp] = [xE_1][xA_1] + [xE_2][xA_2] + [xE_3][xA_3] = \sum_1^3 [xE_i][xA_i].$$

Since x occurs twice in each term on the right side, we may, if we wish, put x^2 outside and fill in the spaces left vacant by a letter l (Lücke: lacuna) so that the quadratic form is written,

$$(5) \qquad [x \cdot xp] = \{[lE_1][lA_1] + [lE_2][lA_2] + [lE_3][lA_3]\}x^2 = \sum_1^3 [lE_i][lA_i]x^2.$$

The factor of x^2 is called a "Lückenform" of the second order; the small letter l in the expression means that the space is to be filled in by a point such as x, y, etc.; l is therefore so to speak "the ghost of a departed point." We denote the lückenform by the letter A_2 and write

$$A_2 = \sum_1^3 [lE_i][lA_i]$$

and, introducing for the A's their values $\sum a_{ik}E_k$, we have

$$A_2 = a_{11}[lE_1]^2 + a_{22}[lE_2]^2 + a_{33}[lE_3]^2 + 2a_{23}[lE_2][lE_3] + 2a_{31}[lE_1][lE_3] + 2a_{12}[lE_1][lE_2]$$

and $[x \cdot xp] = A_2x^2$, so that $A_2x^2 = 0$ becomes the equation of the polecurve. In the same way we have for the bilinear form $[y \cdot xp] = \sum [yE_i][xA_i]$; but, since for any polarity we have the fundamental equation $[y \cdot xp] = [x \cdot yp]$, we have

$$[x \cdot yp] = [y \cdot xp] = \sum_1^3 \frac{[xE_i][yA_i] + [yE_i][xA_i]}{2},$$

so that we may write

$$[x \cdot yp] = [y \cdot xp] = A_2xy = \sum [lE_i][lA_i]\{yx\} = \frac{[xE_i][yA_i] + [yE_i][xA_i]}{2}$$
$$= A_2yx;$$

xy is an algebraic product since $A_2xy = A_2yx$.

It is important to note that each term of (5) is a product of two factors which may be considered as lacunary forms of the first order and the product is not changed by interchanging the factors; the product $[xE_1][xA_1]$, for example, vanishes whenever the point is on the line-pair containing the stäbe E_1 and A_1, and in the case of the quadratic terms the line-pair becomes a double line. We may therefore say that the products in the separate terms of the lückenform (5) represent line-pairs, considered as reducible curves of the second order. In order to simplify the expression we leave out the l's and the brackets; the remaining stab-factors $E_1, A_1, \cdots$ we combine into products E_1A_1, $E_2A_2, \cdots$ of the algebraic kind, since the original products were algebraic just as the product xy. The new expression for the sum of such factors we shall call a *potenzform of the second order* and denote it by the symbol $A^{(2)}$ so that

$$A^{(2)} = \sum_1^3 E_iA_i = a_{11}E_1^2 + a_{22}E_2^2 + a_{33}E_3^2 + 2a_{23}E_3E_2 + 2a_{31}E_3E_1 + 2a_{21}E_2E_1,$$

and this potenzform shall henceforth mean precisely what is expressed by the equation $A_2x^2 = 0$, that is, a curve of the second order. The quantities E_1, ${}^2E_2$, ${}^2E_3$, ${}^2E_2E_3$, E_3E_1, E_1E_2 are *the six units of the second order*. Conversely, the lückenform may easily be constructed, the potenzform being given. Moreover, if we want the curve itself from the potenzform, all we have to do is to substitute for the fundamental stäbe E_1, E_2, E_3 the coordinates ξ_1, ξ_2, ξ_3. Thus the potenzform $a_{23}E_2E_3 + a_{31}E_3E_1 + a_{12}E_1E_2$ represents the curve of the second order $a_{23}\xi_2\xi_3 + a_{31}\xi_3\xi_1 + a_{12}\xi_1\xi_2 = 0$ which is circumscribed about the fundamental triangle; the lückenform is

$$a_{23}[lE_2][lE_3] + a_{31}[lE_3][lE_1] + a_{12}[lE_1][lE_2].$$

In the dual case, starting with a polarity of the second class, we obtain a ternary lückenform of the second class

$$A_2 = [e_1L][a_1L] + [e_2L][a_2L] + [e_3L][a_3L] = \sum [e_iL_i][a_iL]$$

or, introducing the values $a_i = \sum A_{ik}e_k$, $A_{ik} = A_{ki}$,

$$A_2 = A_{11}[e_1L]^2 + A_{22}[e_2L]^2 + A_{33}[e_3L]^2 + 2A_{32}[e_3L][e_2L] + 2A_{31}[e_3L][e_1L] + 2A_{12}[e_2L][e_1L]$$

and the polar curve of the polarity P has the equation $a_2U^2 = 0$. Two lines U and V are conjugate when the bilinear form $a_2UV = 0$. The potenzform corresponding to the lückenform of the second class is now

$$A^{(2)} = A_{11}e_1^2 + A_{22}e_2^2 + A_{33}e_3^2 + 2A_{32}e_3e_2 + 2A_{31}e_3e_1l + 2A_{12}e_1e_2,$$

which represents the curve $a_2U^2 = 0$. The most general potenzform of the second class is derived from the *six units of the second class*, viz.: e_1^2, e_2^2, e_3^2, e_2e_3, e_3e_1, e_1e_2, and the polar curve may be obtained from it by substituting e_i for u_i in the equation of a conic of the second class.

The combinatory product of a potenzform of the second order and a potenzform of the second class is now defined as follows: $[A^{(2)}b^{(2)}] = A_2b^{(2)}$ and $[b^{(2)}A^{(2)}] = b_2A^{(2)}$, that is, we put the combinatory product equal to the product obtained by substituting for the first potenzform the corresponding lückenform and consider the extensive factor of the second potenzform as "filling in" factor intended for the lacunae of the lückenform. The following theorem is then proved: *The combinatory product* $[A^{(2)}b^{(2)}]$ *vanishes when, and only when, the two potenzformen are apolar.* There is however a "fly in the ointment" right here. This elaborate preparation of an algorithm to express the apolarity of two polar systems is not applicable to non-involutary or general reciprocities. We have to go back to the former condition $[rS] = 0$, equation (4).

In the remainder of the chapter special cases of apolarity are given and two theorems of von Staudt on polar triangles and quadrilaterals are proved.

In our journey through 950 pages of this treatise of 1250 pages we have met with the cubic only once, the cubic of involution mentioned above. It is therefore a pleasant relief when the author in the following Chapters XII and XIII allows us to renew our old acquaintances, the curves of the third order and class, although in a stranger garb. A lückenform of the third order is written

$$A_3 = a_{111}[lE_1]^3 + \cdots + 3a_{112}[lE_1]^2[lE_2] + \cdots + a_{123}[lE_2][lE_2][lE_3] + \cdots,$$

and a lückenform of the third class

$$a_3 = A_{111}[e_1L]^3 + \cdots + 3A_{112}[e_1L]^2[e_2L] + \cdots + 3A_{123}[e_1L][e_2L][e_3L] + \cdots,$$

the corresponding curves of the third order and the third class being

$$A_3x^3 = \{ \sum a_{iks}[lE_i][lE_k][lE_s] \} x^3 = 0,$$
$$a_3U^3 = \{ \sum A_{iks}[e_iL][e_kL][e_sL] \} U^3 = 0 .$$

Several familiar theorems on polars of a cubic are then proved. The Hessian of a cubic is expressed by the vanishing of a combinatory product of three lacunary forms $H_3z^3=[A_3ze_1\cdot A_3ze_2\cdot A_3ze_3]$, where $H_3=[A_3le_1\cdot A_3le_2\cdot A_3le_3]$ is a ternary lückenform of the third order. The Hessian of a curve of the third class is represented as a combinatory product of three lückenformen of the third class in the form $h_3W^3=[a_3WE_1\cdot a_3WE_2\cdot a_3WE_3]=0$. The Grassmann point-calculus seems to be well adapted to a careful treatment of the inflectional tangents to a cubic, and the author takes special care in his discussion of the real and imaginary configurations connected with these tangents. The same care is also taken with the dual configuration of the 9 cuspidal tangents to the cubic of the third class.

Chapter XIV deals with nets of conics and their duals, webs of conics. The chapter ends with a section on domains of polarity, and a final section on the Cayleyan curve of a net (web). In the last two chapters metrical properties of conics and their relation to the circular points at infinity are considered. Oblique and rectangular axes are used in turn, and a host of special theorems are produced. The projective properties of some special nets and webs of conics are given due consideration in the last chapter.

As a magisterial work the Projective Geometry of H. Grassmann, Jr. takes a high rank. Although somewhat diffuse at times, it is a marvel of painstaking care and exactitude. It is singularly free from missprints. On page 145, (vol. 2, Part 2) sixth line from bottom, the words "der Ableitszahlen" should be inserted after the word "Funktionen." On p. 284, line 19 from above, the word "und 789" should be inserted after the number "791." On p. 346, last line, "Potenzreihen" should read "Potenzformen."

It is to be regretted that the author had to depart before he could accomplish what was doubtless his intention, an extension of the work to the quaternary field. The notes published by the author in Volume I of Grassmann's Gesammelte Werke, pages 438–464, seem to be a tentative beginning in this direction.

"Hans bok kom ikke, han selv gik derhen,
Hvor tankernes lov ikke skrives med pen."
—B. Björnson

JOHN EIESLAND