ON THE NUMBER OF APPARENT TRIPLE POINTS OF SURFACES IN SPACE OF FOUR DIMENSIONS

BY B. C. WONG

Two hypersurfaces in 4-space of orders μ and ν respectively intersect in a surface F of order $\mu\nu$. F has a certain number H of apparent triple points, that is, lines that can be drawn through a given point meeting F three times. If F degenerates into an F_1 of order m_1 and an F_2 of order m_2 where $m_1+m_2=\mu\nu$, then H is the sum of the numbers h_{30}, h_{21}, h_{12}, h_{03}, where h_{ij} is the number of lines that pass through a given point and meet F_1 i times and F_2 j times. It is the purpose of this paper to determine H and, if F is composite, to determine the distribution of the h_{ij} lines.

The formula for H can be readily obtained by calculating the order of the restricted system of equations resulting from imposing upon two binary equations of orders μ and ν respectively the conditions that they have three common roots.* But this method does not offer a ready means for the determination of the distribution of the h_{ij} lines if F is composite. The following method seems well adapted for the purpose.

Suppose, temporarily, that the two hypersurfaces giving F be composed of μ and ν hyperplanes α_k, β_l $[k=1, 2, \cdots, \mu;$ $l=1, 2, \cdots, \nu]$. Then F is made up of $\mu\nu$ planes $\alpha_k\beta_l$. We construct the rectangular array

$$\text{(A)}\qquad \begin{array}{ccccc} \alpha_1\beta_1 & \alpha_2\beta_1 & \alpha_3\beta_1 & \cdots & \alpha_\mu\beta_1 \\ \alpha_1\beta_2 & \alpha_2\beta_2 & \alpha_3\beta_2 & \cdots & \alpha_\mu\beta_2 \\ \alpha_1\beta_3 & \alpha_2\beta_3 & \alpha_3\beta_3 & \cdots & \alpha_\mu\beta_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_1\beta_\nu & \alpha_2\beta_\nu & \alpha_3\beta_\nu & \cdots & \alpha_\mu\beta_\nu \end{array}$$

* Salmon, *Modern Higher Algebra*, 4th ed., Lesson 19.

and interpret it as the symbolic representation of F, proper or improper. In this interpretation we remove the assumption that the hypersurfaces are composed of hyperplanes and the $\alpha_k\beta_l$ are to be regarded as mere symbols.

Each of the constituents of the array (A), taken alone, represents a plane. A pair of constituents represents a quadric surface or two incident planes if the constituents are in the same row as $\alpha_1\beta_1$, $\alpha_2\beta_1$ or in the same column as $\alpha_1\beta_1$, $\alpha_1\beta_2$; two non-incident planes if they are in different rows and columns from $\alpha_1\beta_1$, $\alpha_2\beta_2$. Three constituents in the same row as $\alpha_1\beta_1$, $\alpha_2\beta_1$, $\alpha_3\eta_1$ or in the same column as $\alpha_1\beta_1$, $\alpha_1\beta_2$, $\alpha_1\beta_3$ represent a cubic surface lying wholly in an S_3: if the constituents are such that one of them lies in the same column with another and in the same row with the third as $\alpha_1\beta_2$, $\alpha_1\beta_1$, $\alpha_2\beta_1$, the cubic surface is a 4-space surface. Three non-incident planes are represented by three constituents all in different rows and columns from $\alpha_1\beta_1$, $\alpha_2\beta_2$, $\alpha_3\beta_3$. Since from a given point only one line can be drawn meeting three non-incident planes each once, the presence of such a triple of constituents, all lying in different rows and columns, in the array means the presence of an apparent triple point on F. The total number of possible triples of this sort in (A) is the total number of possible apparent triple points of F and the formula for this number is evidently

$$(1) \qquad H = \mu\nu(\mu - 1)(\mu - 2)(\nu - 1)(\nu - 2)/6.$$

Now if F is composed of an F_1 of order m_1 and an F_2 of order m_2, the constituents of (A) are divided into two groups: one of m_1 constituents representing F_1 and the other of m_2 constituents representing F_2. Then h_{30} is the number of triples of constituents lying in different rows and columns of the first group; h_{03} the number of similar triples in the second group; h_{21} the number of triples each consisting of a pair of constituents in the first group and one constituent in the second; h_{12} the number of triples each consisting of one in the first and two in the second. Evidently H is the sum of all the h_{ij}, that is

$$(2)\quad (m_1 + m_2)(\mu - 1)(\mu - 2)(\nu - 1)(\nu - 2) = 6(h_{30} + h_{21} + h_{12} + h_{03}).$$

From the very nature of the case we have also

$$(3)\quad \begin{aligned} m_1(\mu - 1)(\mu - 2)(\nu - 1)(\nu - 2) &= a_0 h_{30} + a_1 h_{21} + b h_{11}, \\ m_2(\mu - 1)(\mu - 2)(\nu - 1)(\nu - 2) &= a_0 h_{03} + a_1 h_{12} + b h_{11}, \end{aligned}$$

and

$$(4)\quad (m_1 - m_2)(\mu - 1)(\mu - 2)(\nu - 1)(\nu - 2) = a_0(h_{30} - h_{03}) + a_1(h_{21} - h_{12}),$$

where a_0, a_1 are numerical constants, b is a function of μ and ν, and h_{11} is the order of the cone of lines through a given point meeting F_1 and F_2 each once, or the number of apparent intersections of the sections of F_1 and F_2 by an S_3. The values of a_0, a_1, being independent of μ and ν, can be determined without difficulty. If we put $m_2 = 0$, and consequently $m_1 = \mu\nu$, $h_{03} = h_{21} = h_{12} = 0$ in (4), we have $h_{30} = H$ and $a_0 = 6$. To determine a_1 let F_2 be of order μ, represented by a row of constituents in (A). Then $m_1 = \mu\nu - \mu$, $m_2 = \mu$, $h_{30} = \mu(\mu-1)(\mu-2) \cdot (\nu-1)(\nu-2)(\nu-3)/6$, $h_{21} = \mu(\mu-1)(\mu-2)(\nu-1)(\nu-2)/2$, $h_{03} = h_{12} = 0$. Substituting in (4), we find $a_1 = 2$. To determine b, it is only necessary to make $m_2 = 1$. Then $h_{30} = h_{12} = 0$ and $h_{11} = (\mu-1)(\nu-1)$. Substituting in the second of (3), we obtain $b = (\mu-2)(\nu-2)$. Then (3) and (4) become

$$(5)\quad \begin{aligned} m_1(\mu - 1)(\mu - 2)(\nu - 1)(\nu - 2) &= 6h_{30} + 2h_{21} + (\mu - 2)(\nu - 2)h_{11}, \\ m_2(\mu - 1)(\mu - 2)(\nu - 1)(\nu - 2) &= 6h_{03} + 2h_{12} + (\mu - 2)(\nu - 2)h_{11}, \end{aligned}$$

and

$$(6)\quad (m_1 - m_2)(\mu - 1)(\mu - 2)(\nu - 1)(\nu - 2) = 6(h_{30} - h_{03}) + 2(h_{21} - h_{12}).$$

From (2) and (5) we obtain

$$(7)\quad 2(h_{21} + h_{12}) = (\mu - 2)(\nu - 2)h_{11},$$

$$(8)\quad (m_1 + m_2)(\mu - 1)(\mu - 2)(\nu - 1)(\nu - 2) = 6(h_{30} + h_{03}) + 3(\mu - 2)(\nu - 2)h_{11}.$$

From (5), (7), (8) one can calculate the h's if any two of them are known. From the divided array representing the degenerate F it is not difficult to obtain the values of two of the h's. Take a simple illustration. Let $\mu=\nu=3$ and $m_1=6$, $m_2=3$. If F_1 is symbolized by

$$\begin{array}{ccc} \alpha_1\beta_1 & \alpha_2\beta_1 & \alpha_3\beta_1 \\ \alpha_1\beta_2 & \alpha_2\beta_2 & \cdot \\ \alpha_1\beta_3 & \cdot & \cdot \end{array}$$

and F_2 by

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \alpha_3\beta_2 \\ \cdot & \alpha_2\beta_3 & \alpha_3\beta_3 \end{array}$$

then we have, by inspection, $h_{30}=1$, $h_{03}=0$. Either from the formulas or by further inspection we see that $h_{11}=10$, $h_{21}=4$, $h_{12}=1$.

It is to be added that the method outlined above, applied to r-space, enables us to show that the number of apparent $(r-1)$-fold points of an $(r-2)$-dimensional variety which is the intersection of two hypersurfaces in S_r of orders μ and ν respectively is

$$H_{r-1} = (r-1)!\binom{\mu}{r-1}\binom{\nu}{r-1}.$$

The same process of reasoning yields the following formulas analogous to (5) and (6):

$$\frac{m_1(\mu-1)!(\nu-1)!}{(\mu-r+1)!(\nu-r+1)!} = \sum_{i=0}^{t-1} a_i h_{r-i-1,i} + \sum_{j=1}^{t} b_j h_{jj},$$

$$\frac{m_2(\mu-1)!(\nu-1)!}{(\mu-r+1)!(\nu-r+1)!} = \sum_{i=0}^{t-1} a_i h_{i,r-i-1} + \sum_{j=1}^{t} b_j h_{jj},$$

$$\frac{(m_1-m_2)(\mu-1)!(\nu-1)!}{(\mu-r+1)!(\nu-r+1)!} = \sum_{i=0}^{t-1} a_i (h_{r-i-1,i} - h_{i,r-i-1})$$

where $t=(r-1)/2$ if r is odd and $(r-2)/2$ if r is even. There is no difficulty in calculating

$$a_i = (r-2)!(r-2i-1),$$

but some difficulty is encountered in calculating b_j which are functions of μ and ν. The following are some of their values:

$$\begin{aligned} b_1 &= (\mu-2)!(\nu-2)!/D, \\ b_2 &= 4(\mu-4)!(\nu-4)!/D, \\ b_3 &= 72(\mu-6)!(\nu-6)!/D, \\ b_4 &= 2880(\mu-8)!(\nu-8)!/D, \quad \text{etc.}, \end{aligned}$$

where $D=(\mu-r+1)!(\nu-r+1)!$

For $r=3$, $a_0=2$, $b_1=1$ as is well known.* For $r=4$, $a_0=6$, $\dot{a}_1=2$, $b_1=b=(\mu-2)(\nu-2)$ as we have seen above.

THE UNIVERSITY OF CALIFORNIA

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* Salmon, *Analytic Geometry of Three Dimensions*, 5th ed., vol. 1, pp. 357, 358.