ZERO-FREE REGIONS OF LINEAR PARTIAL FRACTIONS*

BY MORRIS MARDEN†

1. *Introduction.* The object of this paper is to determine simple regions in the plane which do not contain any zeros of the partial fraction

$$\Phi = \sum_{j=1}^{n} \frac{\alpha_j}{z - a_j},$$

when the α_j's are complex constants.

The case of real α_j's has already been adequately treated by Gauss, Lucas, Jensen, and Bôcher in connection with their study of the derivative of a polynomial and the jacobian of two binary forms. In terms of Φ, their results may be stated as follows.

(a) If all the α_j's have the same sign, there are no zeros of Φ outside of any convex polygon enclosing the points a_j. If all the a_j, in addition, lie on the line-segment AB, there are no zeros of Φ except at points on AB‡.

(b) If all the α_j's have the same sign, and the a_j's are either real or in conjugate imaginary pairs, there are no imaginary zeros of Φ at points outside of all the Jensen circles.|| In any Jensen circle containing k a_j's, and not overlapping any other Jensen circle, there are at least $k-1$ and at most $k+1$ roots of Φ.¶

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‡ The result of which this theorem is an immediate consequence was first stated by Gauss, (Werke, vol. 3, p. 112), in 1816, but rediscovered by Lucas (Comptes Rendus, 1868). This theorem, with reference to the derivative of a polynomial, was first stated by Lucas, Journal de l'École Polytechnique, vol. 46 (1879), p. 8.

|| See §6 for definition. This part of the theorem was stated without proof by Jensen, Acta Mathematica, vol. 36 (1912), p. 190; proved by Walsh, Annals of Mathematics, vol. 22 (1920), pp. 128–144.

¶ Walsh, ibid.

(c) If $\sum\alpha_j=0$ and if all the a_j corresponding to positive α_j and to negative α_j lie respectively in the non-overlapping circular regions C_1 and C_2, there are no zeros of Φ at points outside of these regions.*

2. *Method.* In the case of complex α_j's as in the case of the real, the zero-free regions of Φ may be readily obtained when $K(\Phi)$ the conjugate imaginary of Φ, is interpreted as the resultant of all the vectors†

$$f_j = \frac{\beta_j}{\bar{z} - \bar{a}_j} \quad \text{and} \quad g_j = -\frac{i\gamma_j}{\bar{z} - \bar{a}_j}; \qquad (j = 1, 2, \cdots, n).$$

where $\beta_j+i\gamma_j=\alpha_j$. If we set $a_j-z=r_je^{i\theta_j}$, we may write

$$f_j = \frac{\beta_j}{r_j}e^{i(\theta_j+\pi)} \quad \text{and} \quad g_j = \frac{\gamma_j}{r_j}e^{i(\theta_j+\pi/2)}.$$

These equations tell us that, as vectors drawn from z, the first along the line a_jz and the second perpendicular to this line, and having magnitudes contingent only upon r_j and α_j, the f_j and g_j are both independent of choice of axes. A zero-free region of Φ is then one in which the vector

$$K(\Phi) = \sum_{j=1}^{n}(f_j + g_j)$$

or its component in any direction does not vanish.

As the α_j are here complex numbers, it is convenient to regard them as points in a separate α-plane. In this plane it may be assumed without loss of generality, due to the form of Φ, that none of the α_j's lies at the origin. This is equivalent to supposing that, for all j,

$$(1) \qquad |f_j| + |g_j| > 0.$$

Since multiplication of Φ by e^{ki} does not affect its zeros, *any*

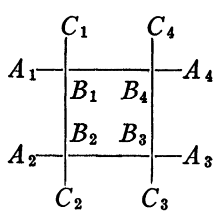
* Bôcher, Proceedings of the American Academy, vol. 40 (1904), pp. 469–484.

† The vector $F_j=f_j+g_j$ may be regarded as the force on a unit particle at z exerted according to the inverse distance law by a particle of mass α_j at a_j. See Bôcher, loc. cit., p. 475, footnote.

theorem true about the zeros of Φ for a given distribution D of the α_j's is likewise true for any distribution D' obtainable from D by a rotation of the α-plane about its origin. Our theorems will however be stated only for simple distributions of the α_j's, the understanding being, of course, that the same theorems hold for somewhat more general distributions.

3. *First Application.* The simplest cases with complex α_j's may be derived from the theorems of §1 by replacing points on the real axis by points on an arbitrary line through the origin. These cases we shall consider, however, by reason of the closing remark of §2, as already accounted for.

Let us assume at first instead that all the points α_j lie in* the first quadrant of the α-plane and the corresponding a_j in* the rectangle $B_1B_2B_3B_4$.



If z is any point in the angle $A_4B_4C_4$, the components of f_k and g_k parallel to A_1A_4 and C_1C_2 will be, respectively,

$$H_k = |f_k| \cos\phi_k + |g_k| \sin\phi_k,$$
$$V_k = |f_k| \sin\phi_k - |g_k| \cos\phi_k,$$

ϕ_k being the angle between f_k and a parallel to A_1A_4.

In order to locate the zeros of Φ in the angle $A_4B_4C_4$, we shall try to find points at which $\sum_1^n H_k = \sum_1^n V_k = 0$.

As $0 \leqq \phi_k \leqq \pi/2$, $H_k \geqq 0$. Because of (1), $H_k = 0$ for all k only in the following cases:

I. $\quad f_k = \phi_k = 0, \qquad$ all k;

II. $\quad g_k = \phi_k - \dfrac{\pi}{2} = 0, \quad$ all k;

III. $\quad f_k = \phi_k = 0$, some k, $g_k = \phi_k - \pi/2 = 0$, remaining k.

In cases I and II, no zeros of Φ turn up because the V_k are of the same sign and therefore $\sum_1^n V_k \neq 0$.

In case III, on the contrary, where some of the a_j lie on

* In §§3, 4, 5, we shall use all our locative phrases "in," "to the right of," etc. in the wide sense, including thereby the boundary points of the regions considered. In §§6 and 7, we shall revert to the strict sense of these terms.

the line B_1B_4 and the corresponding α_j on the imaginary axis, and the remaining a_j lie on the line B_3B_4 and the corresponding α_j on the real axis, there is a possibility of $\sum_1^n V_k=0$ and of a zero of Φ at the point B_4.

Similar considerations hold in the remaining angles $A_kB_kC_k$. *There are no zeros of Φ in any of the angles $A_kB_kC_k$, except perhaps at just one of the points B_k. If, furthermore, all the a_k lie on B_1B_4, there are no zeros to the right of C_3C_4 or to the left of C_1C_2.*

4. *Corollaries.* More elegant results for the case that the α_j's are in the first quadrant can now be obtained if the corresponding α_j are required to lie in a circle C of radius R. In the concentric circle C' of radius $2^{1/2}R$, a square S can be inscribed which will circumscribe C and which, when suitably chosen, will include in an angle $A_kB_kC_k$ any point outside C'. Consequently *there are no zeros of Φ outside of C'.*

If $n=2$, "outside of C'" must be taken in the strict sense. If a_1 and a_2 are situated on C, $\pi R/2$ units apart, S may have a_1 on one side and a_2 on the adjacent side. Case III of §3 would then permit a zero to be on C', as the example

$$\frac{i}{z-i}+\frac{1}{z-1}=0$$

with the unit circle as C, shows.

If the points a_j just considered are all on the line-segment AB, Φ will have a larger zero-free region. On AB as diagonal a rectangle R can so be constructed as to include in an angle $A_kB_kC_k$ any point outside of the circle Γ having AB as diameter. *Outside of Γ there are therefore no zeros of Φ.*

Again "outside of Γ" must be given a strict meaning when $n=2$. If a_1 and a_2 are at A and B, case III arises with its possibility of a zero on Γ, as for instance in the example

$$\frac{1}{z-1}+\frac{i}{z+1}=0,$$

where A: $(-1, 0)$ and B: $(1, 0)$.

5. *Other Applications.* By essentially the methods of §3, the following results may also be deduced:

(a) *Suppose the α_j to consist of two groups G_1 and G_2, the former situated in the first quadrant and the latter in the second quadrant of the α-plane. If the a_j corresponding to G_1 and G_2 lie respectively in the angles $A_1B_1C_2$ and $A_4B_4C_3$, there are no zeros of Φ in the strip $C_1B_1B_4C_4$, save perhaps upon the segment B_1B_4.*

The exceptions may occur only if $G_1 = G_{1a} + G_{1c}$ and $G_2 = G_{2a} + G_{2c}$, where G_{1a} and G_{2a} have their α_j's on the imaginary axis, G_{1c} and G_{2c} their α_j's on the real axis, and G_{1a}, G_{1c}, G_{2a}, and G_{2c} have their a_j's respectively on the segments A_1B_1, B_1C_2, A_4B_4, B_4C_3.

If $G_{1c} = G_{2c} = 0$ (that is, G_{1c} and G_{2c} contain no points), any point of B_1B_4 may be a zero of Φ.

If $G_{1a} \cdot G_{1c} \neq 0$ and $G_{2c} = 0$, the point B_1 is the only possible zero of Φ on B_1B_4.

If $G_{1c} = 0$ and $G_{2a} \cdot G_{2c} \neq 0$, the point B_4 is the only possible zero of Φ on B_1B_4.

If the lines C_1C_2 and C_3C_4 are coincident, we shall exclude the case $G_{1a} = G_{2a} = 0$, but include the cases

$$G_{1a} = 0 \text{ and } G_{2a} \cdot G_{2c} = 0,$$

$$G_{2a} = 0 \text{ and } G_{1a} \cdot G_{1c} = 0,$$

and

$$G_{1a} \cdot G_{1c} \cdot G_{2a} \cdot C_{2c} \neq 0,$$

in which cases B_1 is the only possible zero of Φ on B_1B_4.

(b) *Suppose the α_j to consist of four groups $G_k (k = 1, 2, 3, 4)$, the group G_k being in the kth quadrant of the α-plane. If the a_j corresponding to G_k lie in the angle $A_kB_kC_k$, there are no zeros of Φ in the rectangle $B_1B_2B_3B_4$.*

We are supposing no two sides of $B_1B_2B_3B_4$ to coincide. A degenerate case of interest is, however, one in which A_1A_4 and A_2A_3 are coincident and the a_j corresponding to G_k lie on A_1B_1 if $k = 1$ or 2, and on A_4B_4 if $k = 3$ or 4. Then, if not all the α_j's lie on the real axis, there are no zeros of Φ on the segment B_1B_4. The theorems of (a) and (b) also hold for $n = \infty$ provided Φ is absolutely convergent in the above-mentioned zero-free regions.

6. *A Jensen Case.* So far no restrictions have been placed upon the behavior of Φ for real values of z. In this section we shall assume, in order to make Φ real on the real axis, that the a_j and the corresponding α_j are both real or both in conjugate imaginary pairs. We shall also suppose the α_j corresponding to an a_j in the upper half-plane also to be in the upper half-plane, and vice-versa. Furthermore, we shall limit the α_j to the first and fourth quadrants of the α-plane.*

Let us construct in the z-plane the circles (the Jensen circles) having as diameters the line segments joining pairs of conjugate a_j, and the indefinite lines (the Jensen lines) through the pairs of conjugate a_j. If we denote by R the region consisting of all points simultaneously outside of all the Jensen circles and to the left of all the Jensen lines, we may define R by the inequalities

$$(x - b_j)^2 + y^2 - c_j^2 > 0, \quad x - b_j < 0\,;$$

all j corresponding to imaginary $a_j = b_j + ic_j$.

In the sum Φ let us discern two types of terms, one corresponding to real a_j:

$$t_1 = \frac{\beta_j}{z - b_j} = \frac{\beta_j[(x - b_j) - iy]}{(x - b_j)^2 + y^2}, \qquad (\alpha_j = \beta_j + \gamma_j),$$

and the other corresponding to a pair of conjugate a_j:

$$t_2 = \frac{\beta_j + i\gamma_j}{z - (b_j + ic_j)} + \frac{\beta_j - i\gamma_j}{z - (b_j - ic_j)}\,.$$

The latter has as component parallel to the y-axis:

$$I(t_2) = \frac{-\,2y[\beta_j\{(x - b_j)^2 + y^2 - c_j^2\} - 2c_j\gamma_j(x - b_j)]}{\{(x - b_j)^2 + (y - c_j)^2\}\{(x - b_j)^2 + (y + c_j)^2\}}\,.$$

As $\beta_j \geqq 0$, $c > 0$, and $\gamma > 0$, we find in R

$$\operatorname{sgn} I(t_1) = \operatorname{sgn} I(t_2) = -\operatorname{sgn} y.$$

The components of t_1 and t_2 parallel to the y-axis will there-

* See the footnote to §3.

fore not vanish at points in R off the real axis. That is to say, *there are no imaginary roots of* Φ *in* R.

The question of the real roots of Φ in R may be easily settled through a study of the real function Φ on the real axis. Again we shall separate the terms corresponding to real a_j from those corresponding to pairs of conjugate a_j. The first

$$S_1 = \frac{\beta_j}{x - b_j} \quad \text{has} \quad \frac{dS_1}{dx} = -\frac{\beta_j}{(x - b_j)^2};$$

the second

$$S_2 = \frac{\beta_j + i\gamma_j}{x - (b_j + ic_j)} + \frac{\beta_j - i\gamma_j}{x - (b_j - ic_j)},$$

$$\frac{dS_2}{dx} = \frac{2\beta_j\{c_j^2 - (x - b_j)^2\} + 4c_j\gamma_j(x - b_j)}{\{(x - b_j)^2 + c_j^2\}^2}.$$

Therefore $d\Phi/dx < 0$ in R. In addition, as x moves to the right from one a_j to the next a_j, Φ changes from $+\infty$ to $-\infty$. In other words, any interval of the real axis in R not containing any a_j has at most one root of Φ; between two successive a_j in R, there is exactly one root of Φ. In short, *if the number of* a_j *in* R *is* k, *the number of real roots of* Φ *in* R *is* k, $k-1$, *or* $k+1$.*

The above equations also allow us to deduce at once the results for the case in which the α_j corresponding to real a_j lie on the positive real axis, and the remaining α_j are in the second and third quadrants of the α-plane. The above theorems will hold for this case if R in these theorems is replaced by K, the region consisting of all points simultaneously inside of all the Jensen circles and to the left of all the Jensen lines. (K may not contain any points.)

These theorems will also be valid when $n = \infty$ provided Φ converges uniformly in any closed region in R or K not including any points a_j, and all the imaginary a_j lie to the right of a fixed line parallel to the y-axis.

* E. B. Van Vleck has an unpublished proof of Jensen's theorem which uses similar analysis.

7. *The Function* $\eta(z)$. As a final application particularly of the methods of §6, we turn to the function

$$\eta(z) = \frac{1}{z}\left\{\frac{\zeta'(z)}{\zeta(z)} - (1 + \log 2\pi)\right\}$$
$$= \frac{2}{z^2} - \frac{1}{z-1} - \sum_{k=1}^{\infty} \frac{1}{k(z+2k)} + \sum_{-\infty}^{\infty} \frac{1}{\rho_n(z-\rho_n)},$$

where $\zeta(z)$ is the Riemann zeta function, and the ρ_n its zeros.

Concerning the ρ_n, it is known* that

$$\rho_{-n} = \bar{\rho}_n$$
$$0 < R(\rho_n) < 1, \qquad \text{(all } n\text{)},$$
$$R(\rho_n) = \tfrac{1}{2}, \qquad \text{for } |I(\rho_n)| \leqq 300,$$

and that $\rho = \frac{1}{2} \pm Ki$, $K = 14.13472(5)$, are the nearest ρ_n to the real axis.

Let us draw a circle of radius K, center at $(1/2, 0)$ and denote by S the interior of this circle to the left of the imaginary axis. Then in S there are no imaginary zeros of $\eta(z)$. In each of the intervals $(0, -2)$, $(-2, -4)$, $(-4, -6)$, $(-6, -8)$, $(-8, -10)$, and $(-10, -12)$ of the real axis there is exactly one zero, and in the interval $(-12, \frac{1}{2} - K)$ there is at most one zero of $\eta(z)$.

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* See Hutchinson, Transactions of this Society, vol. 27 (1925), p. 49, and references therein. My attention was called to the logarithmic derivative of the zeta function by C. E. Hille.