

CLASSES OF DIOPHANTINE EQUATIONS WHOSE
POSITIVE INTEGRAL SOLUTIONS
ARE BOUNDED*

BY D. R. CURTISS

1. *Introduction.* If upper bounds have been found for the positive integral solutions of a diophantine equation, the problem of obtaining all such solutions is reduced to making a finite number of trials. It may therefore be of interest to note certain cases where upper bounds are given by simple algebraic processes. Hereafter the term *solution* will always mean a solution in positive integers.

Our starting point is the observation that if $P(t)$ is a polynomial in t , then all positive values of x that satisfy the inequality $P(1/x) \geq 0$ are bounded if (and only if) the term of lowest degree in $P(t)$ has a negative coefficient.

2. *A Type whose Solutions are always Bounded.* Every algebraic diophantine equation in n variables x_1, x_2, \dots, x_n can be thrown into a form where the right side is zero and the left side is a polynomial in the reciprocals of the x 's. When this has been done, the first type here to be considered is the following:

$$(1) \quad F\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}\right) - k = 0,$$

where F is a polynomial all of whose coefficients are positive, while k is a positive constant, and $F(0, 0, \dots, 0) = 0$.

The positive integral solutions of every equation of type (1) are bounded.

To prove this statement, and to show how to obtain bounds for the solutions, let us first consider a solution such that $x_1 \leq x_2 \leq \dots \leq x_n$. We shall then have

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$$(2) \quad F\left(\frac{1}{x_1}, \frac{1}{x_1}, \dots, \frac{1}{x_1}\right) - k \geq 0,$$

since each term of F is not decreased when another x is replaced by x_1 . The term of lowest degree in (2), $-k$, is negative, hence, from the observation made in the second paragraph of this paper, x_1 must be bounded. To obtain an explicit upper bound we note that, since $1/x_1^n \leq 1/x_1$ when n and x_1 are positive integers, and since all the terms of F are positive, we have

$$F\left(\frac{1}{x_1}, \frac{1}{x_1}, \dots, \frac{1}{x_1}\right) \leq \frac{1}{x_1} F(1, 1, \dots, 1).$$

Hence when this result is applied to (2) we obtain

$$\frac{1}{x_1} F(1, 1, \dots, 1) - k \geq 0,$$

or

$$(3) \quad x_1 \leq \frac{1}{k} F(1, 1, \dots, 1).$$

Usually a lower bound than this can be derived by finding a closer approximation for the (unique) positive root of the equation obtained from (2) by retaining only the sign of equality.

Let us now find a bound for x_r ($r \leq n$) when upper bounds X_j have been assigned for each x_j from $j=1$ to $j=r-1$ in a solution where the x 's form an ascending sequence from x_1 to x_n . We write (1) in the form

$$(4) \quad \begin{aligned} & F\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}\right) \\ & - F\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_{r-1}}, 0, 0, \dots, 0\right) \\ & = k - F\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_{r-1}}, 0, 0, \dots, 0\right). \end{aligned}$$

The first two lines of (4) reduce to a polynomial

$$F_r\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_{r-1}}, \frac{1}{x_r}, \frac{1}{x_{r+1}}, \dots, \frac{1}{x_n}\right)$$

each of whose terms is positive and involves at least one of the variables x_r, \dots, x_n . Its value will not be decreased if each of the variables x_{r+1}, \dots, x_n is replaced by x_r . The last line of (4) is positive for all x 's from x_1 to x_{r-1} that belong to solution systems, since the F function that appears here consists only of certain terms of the complete F in equation (1) and must lack some of the terms* of the complete F when $r-1 < n$. Hence the inequality derived from (4),

$$F_r\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_{r-1}}, \frac{1}{x_r}, \frac{1}{x_r}, \dots, \frac{1}{x_r}\right) - \left[k - F\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_{r-1}}, 0, 0, \dots, 0\right) \right] \geq 0,$$

is of the type $P(1/x_r) \geq 0$, with term of lowest degree in $1/x_r$ negative. It follows that x_r is bounded for each set of values of the preceding x 's. If these preceding x 's are bounded, there must be an upper bound for all the values of x_r that belong to solution systems. Since we have shown that x_1 is bounded, it follows that all the x 's are bounded, in solutions where $x_1 \leq x_2 \leq \dots \leq x_n$. We conclude at once that they are bounded for every order of relative magnitudes.

If the x 's are arranged in order of magnitude from x_1 to x_n , an explicit upper bound for x_r , of which that given by (3) for x_1 is a special case, is indicated by the inequality

$$(5) \quad x_r \leq \frac{1}{m_r} F_r(1, 1, \dots, 1) = X_r,$$

where m_r is the least positive value of

$$k - F(1/x_1, 1/x_2, \dots, 1/x_{r-1}, 0, 0, \dots, 0)$$

for $x_j \leq X_j$, ($j = 1, 2, \dots, r-1$). Since m_r may be difficult to

* We assume that each of the variables x_1, \dots, x_n is explicitly present in the F function of equation (1).

evaluate, we note that (5) can be replaced by the weaker inequality

$$(6) \quad x_r \leq D_r F_r(1, 1, \dots, 1) = X'_r,$$

where D_r is the product of all the denominators of terms of

$$k - F(1/X'_1, 1/X'_2, \dots, 1/X'_{r-1}, 0, 0, \dots, 0).$$

As an illustration, consider the equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} = 1.$$

Here $F(1, 1, \dots, 1) = n$, and $F_r(1, 1, \dots, 1) = n - r + 1$. The value of $1/m_r$ is u_r , where $u_1 = 1$, $u_{k+1} = u_k(u_k + 1)$.* Thus from (5) we obtain the upper bounds

$$X_r = (n - r + 1)u_r, \quad (r = 1, 2, \dots, n).$$

From (6) we have another set of bounds X'_r , such that

$$X'_r = (n - r + 1)X'_1 X'_2 \dots X'_{r-1}.$$

Hence

$$X'_1 = n, \quad X'_2 = (n - 1)n,$$

$$X'_r = (n - r + 1)(n - r + 2)(n - r + 3)(n - r + 4)^2 \dots n^{r-2}, \quad r > 1.$$

3. *More General Types.* Equations of more general type than (1) can be written in the form

$$(7) \quad F\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}\right) = G\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}\right),$$

where F and G are polynomials all of whose coefficients are positive; we suppose all possible cancellations to have been performed. One or more of the variables may not be present in F , and the same may be true of G , but no variable is to be absent from $F - G$. We now investigate bounds for solutions where $x_1 \leq x_2 \leq \dots \leq x_n$.

If there is a constant term on either side of (7), x_1 is bounded.

* See *On Kellogg's diophantine problem*, American Mathematical Monthly, vol. 29 (1922), p. 380.

For, if, for example, we have $G(0, 0, \dots, 0) = k \neq 0$, then $F(1/x_1, 1/x_1, \dots, 1/x_1) \geq k$, and formula (3) gives a bound for x_1 . If (7) has no constant term, we consider the two inequalities derived from (7),

$$\begin{aligned} F\left(\frac{1}{x_1}, \frac{1}{x_1}, \dots, \frac{1}{x_1}\right) - G\left(\frac{1}{x_1}, 0, \dots, 0\right) &\geq 0, \\ G\left(\frac{1}{x_1}, \frac{1}{x_1}, \dots, \frac{1}{x_1}\right) - F\left(\frac{1}{x_1}, 0, \dots, 0\right) &\geq 0. \end{aligned}$$

If the term of lowest degree in $1/x_1$ on the left of either of these inequalities is negative, we conclude that x_1 is bounded.

Again, using the notation of the earlier part of this paper, we deduce from (7) the inequality

$$\begin{aligned} (8) \quad &F_r\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_{r-1}}, \frac{1}{x_r}, \frac{1}{x_r}, \dots, \frac{1}{x_r}\right) \\ &- \left[G\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_{r-1}}, 0, 0, \dots, 0\right) \right. \\ &\left. - F\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_{r-1}}, 0, 0, \dots, 0\right) \right] \geq 0, \end{aligned}$$

and another in which F_r is replaced by G_r , and F and G are interchanged. Unless the expression in brackets is zero, one of these inequalities is of the type $P(1/x_r) \geq 0$, with negative constant term. Hence *unless the pair of equations*

$$\begin{aligned} (9) \quad &F\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}\right) = G\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}\right), \\ &F\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_{r-1}}, 0, 0, \dots, 0\right) \\ &= G\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_{r-1}}, 0, 0, \dots, 0\right) \end{aligned}$$

has a solution in positive integers x_1, x_2, \dots, x_n , arranged in that order of magnitude, every solution of (7) so ordered has x_r bounded if all the preceding x 's are bounded. From this we at

once obtain a sufficient condition that all the x 's be bounded.

If the above condition fails we may replace the expression in brackets in (8) by

$$G\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_r}, 0, 0, \dots, 0\right) - F\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_r}, 0, 0, \dots, 0\right),$$

and F_r by F_{r+1} . If in the new inequality thus obtained the term of lowest degree in $1/x_r$ is negative, or if it is negative in the companion inequality, we infer again that x_r is bounded if this is true of the preceding x 's.

An example to show that the above conditions may not be fulfilled is given by the equation

$$1 + \frac{1}{x_2} = \frac{1}{x_1} + \frac{1}{x_3^2}.$$

Here for $r = 2$ the second equation of system (9) becomes

$$1 = \frac{1}{x_1},$$

and the pair of equations (9) is satisfied by $x_1 = 1$, $x_2 = x_3^2$, where x_2 and x_3 are not bounded. On the other hand, many equations whose solutions are bounded escape these tests; an example is

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_1} - \frac{3}{x_2} = 0,$$

whose only solution in positive integers is $x_1 = x_2 = 1$.

4. *Algebraic Equations with Positive Integral Roots.* A corollary of the first theorem of this paper concerns itself with algebraic equations

$$x^n - a_1x^{n-1} + a_2x^{n-2} - \dots + (-1)^na_n = 0,$$

all of whose roots are positive integers. There is, of course, an infinite number of such equations for any given n . But

there is only a finite number whose coefficients satisfy a relation

$$F\left(\frac{a_1}{a_n}, \frac{a_2}{a_n}, \dots, \frac{a_{n-1}}{a_n}\right) = k,$$

where F is a polynomial with positive coefficients and $k > 0$; for F is a polynomial in the reciprocals of the roots, and, when thus expressed, F has no constant term, so that the first theorem of this paper applies. We could obtain upper bounds for the roots, and therefore for the a 's, by the methods of this paper. For example, if $a_{n-1} = a_n$, and if x_1, x_2, \dots, x_n are the roots, the x 's must be solutions of the equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} = 1,$$

which has been discussed in § 2.

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ERRORS IN KRAITCHIK'S TABLE OF LINEAR FORMS

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Tables of the linear forms that belong to a given quadratic residue D , or in other words, the linear divisors of the quadratic form $t^2 - Du^2$ were first published by Legendre.* A list of errors in these fundamental tables has been given by D. N. Lehmer.† Kraitchik‡ has recalculated and extended these tables to the limit $D = \pm 250$. It is of great importance in using the table that every entry be correct. Therefore in constructing his factor stencils, D. N. Lehmer found it advisable to make a new table by means of a more or less graphical method.§ This table which has not been pub-

* *Théorie des Nombres*, 1st. ed., Tables III-VII, 1798.

† This Bulletin, vol. 8 (1902), p. 401. See also the correction in this Bulletin, vol. 31 (1925), p. 228.

‡ *Théorie des Nombres*, vol. 1, p. 164-186, Paris, 1922. *Recherches sur la Théorie des Nombres*, vol. 1, p. 205-215, Paris, 1924.

§ This Bulletin, vol. 31 (1925), pp. 497-498.