KÖNIG AND KRAFFT'S ELLIPTIC FUNCTIONS

Elliptische Funktionen. By R. König and M. Krafft. Leipzig, Walter de Gruyter and Co., 1928. 263 pp.

Many are the methods of approach to the study of elliptic functions. Since Abel's classic memoir (1826), an outgrowth of his researches on the lemniscate definite integral, much attention has been given to the development of the theory of algebraic and of abelian functions and of the chapter in this theory which has to do with the subject of this book. Some, as Riemann (1857) and Weierstrass (1875–6) followed first the lead of Abel in the 1826 paper (anticipated by Gauss as early as 1798) but later redeveloped the theory using the algebraic differential equation of the first order as a cornerstone on which to build. Others, as Jacobi, (1829 and 1834), and Eisenstein, starting with properties of multiply periodic functions, have developed the theory of elliptic functions as an elegant chapter in the theory of functions. The geometric properties and applications interested Riemann, Clebsch, (1864), and Brill and Noether, (1874); while the latest and indeed the least traveled avenue of approach, viz., the number-theoretic, called also arithmetic, was employed by Dedekind and Weber, (1879), by Kronecker, (1881), and by Hensel and Landsberg, (1902).

This book by König and Krafft gives the first detailed exposition of the arithmetic theory as applied specifically to functions having the three fundamental properties which characterize elliptic functions, viz., (1) they form a linear manifold, (2) they form an algebraic field, (3) they are of genus one. The aim has been "not to set up theorems and formulas with lemmas and computations constructed for the purpose but to develop the results organically and genetically" from these three properties of which the first is the simplest and at the same time the most general. Thus it is evident that the scope of the book is considerably more than the development of the arithmetic theory of functions of genus one. As the introduction tells us: The reader becomes familiar with Riemann's doctrine, which lies at the bottom of the whole plan of the book, a doctrine which not only supplies a Riemann surface on which the function is single-valued, but which also enables the reader to understand the classification of elliptic functions as a member of a great organism, Riemann function-pairs as an instance of Riemann function-systems; moreover, according to Riemann and F. Klein, the fundamental law which rules the organism is a law of duality by which every theorem on functions can be restated for differentials. The reviewer thinks the book justified by their claim that "though the material is not new, the contribution is in the painstaking methodical simplifications made even in the things well known." It is a contribution which is welcome. The book can be read with pleasure by one who is familiar with the theory of elliptic functions in any of its phases or ramifications, and it can be used quite satisfactorily as a text with a class which knows the elements of function theory though not without considerable amplification of the material by the instructor. In particular the great array of notational symbols in the book makes the reading more difficult than the mathematical theory should require. If it could have been presented without this piling of symbol upon symbol it would have been better.

After reading the first three chapters a reader can turn to the general theory of algebraic functions as given in Hensel and Landsberg's treatise and in Chapter 24 of the third volume of Weber's *Lehrbuch der Algebra* and get the extension to an n-valued algebraic function with considerable ease. One should not assume that because the elliptic case is a special instance of the general case it is not worth while to study it in as detailed a fashion as is done here. As a matter of fact the translation of the notions from general (p arbitrary) to special ($p=1$) are far from obvious while the wide applications of the elliptic case make the special case important per se.

The list of chapter headings gives an indication of the scope of the book:

1. Grundlagen.
2. Arithmetischer Teil.
3. Die Elementarfunktionen.
4. Elementarfunktionen und Elementardifferentiale von zwei Veränderlichen.
5. Die Elliptischen Integrale.
6. Funktionen zweiter und dritter Art.
7. Das Abel'sche Theorem.
8. Konforme Abbildung durch elliptische Funktionen und birationale Transformationen.
9. Die elliptischen Funktionen und Integrale im Periodenparallelogramm.
10. Die Funktionen des Ringes.

In the first chapter are given fundamental definitions and notations. An elliptic algebraic function is single-valued and regular save for a finite number of poles on a two sheeted Riemann surface with four branch points. It is a special case of a class (K) of (Riemann) function-pairs on two sheets on which the branch cuts have been made but not joined. All properties of elliptic functions as fast as developed are discussed with respect to elliptic differentials.

Chapter 2 gives the arithmetic part. Using the methods of number theory we can find a basis for the entire class; an ideal basis for the ideal of the divisor, Q; a multiple basis for the divisor, Q, at all finite points; all functions which are multiples save possibly at P_∞ of a given divisor; a normal basis for such functions; and finally a basis which is normal at P_∞.

Chapter 3 treats of the characterization of elliptic functions and differentials. An elementary function with a fixed pole of arbitrary order is the element in terms of which any elliptic function is expressible. This elementary function plays a rôle similar to the fraction $1/(x-a)$ in the decomposition of rational functions into partial fractions. A corresponding elementary differential is defined. The method is extended to functions of two complex variables in Chapter 4, while in Chapters 5 and 6 elliptic integrals on the Riemann surface which has been made simply connected by $2p$ period cuts are constructed in terms of integrals of the elementary differentials of two complex variables. Here the Weierstrass, Klein, and Prym prime functions are defined, again in terms of the elementary functions.

Chapter 7 gives an elegant exposition of the theory underlying Abel's theorem for elliptic integrals of the three kinds. In these seven chapters the functions have been studied on the two-sheeted Riemann surface.

In Chapter 9 the functions are studied on the period parallelogram which in its turn in Chapter 10 is mapped conformally on a ring. This follows an excellent discussion of conformal mapping by elliptic functions and birational transformations in Chapter 8. The arithmetic method is abandoned now for the more usual methods of analysis. The authors tell us that the shriveling of the theory due to the fact that now functions and differentials become identical gives greater simplicity but with loss in beauty and richness of structure. The Fourier developments of elliptic functions are obtained directly from known properties of integrals on the ring and may then be carried back to the period parallelogram. The simplicity of this exposition will appeal to those who know how Fricke obtained the same results by the application of functional equations.

The work of printer and editor have been well done. The text is almost entirely free from typographical errors.

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