BATCHELDER ON DIFFERENCE EQUATIONS

An Introduction to Linear Difference Equations. By Paul M. Batchelder. Published with the cooperation of the National Research Council. Cambridge, Harvard University Press, 1927. 209 pp.

The subject of difference equations is but one of the divisions of the rather extensive field of the calculus of finite differences. The other usually recognized divisions are interpolation, summation of series, and mechanical quadrature, and these topics, together with difference equations, were treated in the classic book by Boole, quite as if they formed an inseparable whole. But within recent years difference equations have received new attention, and the object of the later studies, and the whole spirit that pervades them, are so different from that of the old, that one is not surprised to observe a tendency to separate the larger subject. Such a separation is usually the characteristic of a growing study.

The work of Boole was what is customarily described as largely formal, and such a quality seems inherent in interpolation and mechanical quadrature. The revolt against pure formalism that has come with the function theory may have led some mathematicians to regard the virtues of these subjects as inferior ones, and their connection with calculation might tend to increase the disfavor in the minds of some. Such persons should experience some of the pleasure of the reformer at the rescue of difference equations from its old associates.

Though the subject of difference equations has been redeemed and has taken on the manners of the modern mathematical world, it has not enjoyed even a season of popularity. It may have been welcomed but it has been given no enthusiastic acclaim, for it has won few intimates and scarcely more acquaintances.

There are certainly merits in the new study, that perhaps may be described best by referring to the gamma function, which has long been so important. It was one of the first functions to be defined by a definite integral, and if it had done nothing more than generalize for positive reals the factorial, it would have been significant. But with the advent of the complex variable the function became more important, for the integral of Euler has meaning everywhere to the right of the axis of imaginaries, and the relation $f(x+1)=xf(x)$ continues the function to the left and defines it except at the points $0,-1,-2,\cdots$, where there are poles. Finally, by using a contour integral, Heine gave a representation uniformly valid where the function has a meaning. The behavior of the function in the distant parts of the plane is very notable, and the series of Stirling was among the first, and is still perhaps the best known, asymptotic series. It would of course be only natural to believe that the origin of all the interesting and peculiar analytic properties of the function lay in the simple difference equation $f(x+1)=xf(x)$ which it satisfies. But Hölder's proof that the gamma function satisfies no algebraic differential equation gave certainty to this belief, and helped to emphasize the importance of a general study of difference equations. As has been pointed out in several places the first distinct contributions were by Nörlund, Galbrun, Carmichael, and Birkhoff, each using a different method.

One cannot help inquiring as to why the field has remained one of restricted interest. It may be due in part to the fact that as soon as one starts to generalize the problems become complex, and one cannot go far without the expense of great pains. This is conspicuous for instance in the recent study of certain "irregular cases" by C. R. Adams, who begins by giving series that formally satisfy the equation, and adds that such a property "can be verified directly, although the labor involved is not inconsiderable."* Such a remark will not cause a person familiar with the field to turn aside, but it is not likely on the other hand that it will make such appeal to the casual eye as will arouse either interest or curiosity. It is not implied by these remarks that difficulty is a hindrance to mathematical study, or that only the simple can be popular, for difficulty has always been a stimulus to study. But if a subject is to enjoy general favor it must give something more than the satisfaction that comes from overcoming formidable obstacles; there must be some other reward than that; there must be results with charm and a compelling interest of their own, or results that, perhaps unexpectedly, throw light upon another problem. And difficulties too must occur mingled with what is easy, for the simple but significant is not only refreshing, but it is for it that one makes the endless search. There is in the combination of what is difficult and what is easy, something that piques one, and haunts the mind with the thought that even the apparent difficulty may be only a stubborn mask. The real triumph does not lie in the mere discovery of truth, but in making it seem simple and natural, so that one feels comfortable and at home in its presence. Up to the present time it seems that the subject of difference equations has lacked such qualities, as well as any profusion of special problems that would mitigate the severity of the necessary but rather cold existence theorems.

The book by Batchelder seeks to introduce the reader to the subject of difference equations without bewildering him with all the intricacies of the general case. After certain preliminaries are disposed of, which include a consideration of asymptotic representation, the book treats the first-order equation with some completeness, the gamma function appearing as a special case. The hypergeometric equation is taken as a sufficiently representative equation of higher order, and one particularly interesting on account of the analogies with the well known differential equation. The development is along the line of Birkhoff's theory. An extensive treatment is given of the so-called irregular cases. Such cases had not been treated before and this part of the book constitutes the writer's own contribution. Even for the special equation the problem is involved.

It seems to the reviewer a fault to all but ignore the purely formal part of the subject. There are in the earlier part of the book some half dozen problems similar to the easier differential equations upon which one first practices. There could well have been a more liberal selection, for such problems not only serve to arouse the interest, but they contribute a great deal towards making one feel

* *On the irregular cases of the linear ordinary difference equation*, Transactions of this Society, vol. 30 (1928), pp. 507–541.

at home with new ideas, and when they are wisely chosen they can exhibit the variety that exists in a subject and suggest new lines of thought. Even simpler problems are not to be despised, for it is not a base thing to get pleasure from manipulating a problem into a practicable answer. One can never be sure that the most uncompromising devotee of existence theorems does not some times seek amusement in such indulgence.

All persons who have given any attention at all to difference equations must hope that the subject will find an appropriate place in the mathematical structure. It would seem that extensive treatises on analysis should devote some space to it, at least to the extent of treating the gamma function in a way different from the traditional one, and going a little beyond the gamma function. It is to be hoped that the material in Batchelder's book may help to make such treatments possible as well as stimulate further study of the subject.

K. P. WILLIAMS

HELLINGER-TOEPLITZ ON INTEGRAL EQUATIONS

Integralgleichungen und Gleichungen mit unendlichvielen Unbekannten. By E. Hellinger and O. Toeplitz. Reprint from the Encyclopädie der Mathematischen Wissenschaften with the addition of a preface by E. Hilb and of a special subject-index. Leipzig-Berlin, Teubner, 1928. Pp. 1335–1616.

In his preface Hilb points out that the book under review appears after a quarter of a century of research in the field of the theory of integral equations. It must be considered, therefore, as a survey of results obtained, and as an account of problems which remain still unsolved. "During several years of cooperative work the authors scrutinized the whole literature as to methods, results and their comparative range." We agree with Hilb that this "Report is indispensable to anybody who desires to penetrate deeply into this subject so extraordinarily important for its applications." An attentive reader, even well versed in the subject, will find many novel features in treating old and new questions, features which are extremely illuminating and inspiring; he will welcome the successful efforts of the authors to unify the multitude of existing methods and to present these methods as parts of a harmonious whole. As notable instances of this kind we may mention the treatment of completely continuous forms (pp. 1403–1413); of normal matrices (p. 1562; this seems to be a new notion introduced by the authors and proving to be quite useful in a number of recent investigations); of symmetrizable kernels (pp. 1536–1543) and matrices (pp. 1563–1575); of a general principle which could be designated as a "principle of preservation of the type" of kernels or matrices (pp. 1391–1392, 1431–1433). It is undoubtedly a good idea (although a departure from the usual style of the Encyklopädie) to give *proofs* of some facts of fundamental importance; the reader will be also pleasantly surprised to find references to some *facts not previously published* (Toeplitz, p. 1573; Toeplitz-Schmidt, p. 1575; Szász, p. 1522). It is hardly necessary to mention that the bibliography of the Report is extremely rich and shows that the authors have canvassed