HILBERT AND ACKERMANN ON MATHEMATICAL LOGIC

Grundzüge der theoretischen Logik. By D. Hilbert and W. Ackermann. Berlin, Julius Springer, 1928. 120 pp.

This book deals with mathematical logic very much after the fashion of the first volume of *Principia Mathematica*. The authors begin with a treatment of the theory of elementary functions of propositions, and go on, in their second chapter, to a consideration of the logic of classes and its application to the traditional syllogism; they then take up a discussion of general propositions that involve a use of the notions "some" and "all" as applied to variables denoting individuals; and finally, in their last chapter, they consider propositions involving generalization with respect to functions, in connection with which a discussion of the paradoxes, the theory of types, and the axiom of reducibility, is entailed. These are, of course, all well known topics; but there are certain features of the book that are peculiar to it, and it is to some of these features that we shall direct attention.

The authors begin their formal analysis of elementary functions (p. 22) with the four primitive propositions:

[a] $(p):p \vee p \cdot \supset \cdot p,$

[b] $(p, q):p \cdot \supset \cdot p \vee q,$

[c] $(p, q):p \vee q \cdot \supset \cdot q \vee p,$

[d] $(p, q, r):\cdot p \supset q \cdot \supset :r \vee p \cdot \supset \cdot r \vee q,$

which are identical with four of the five propositions employed in *Principia Mathematica*. They have also a rule of substitution and a rule of inference; but they omit one proposition used by Whitehead and Russell, namely $(p, q, r):\cdot p \cdot \vee \cdot q \vee r : \supset :q \cdot \vee \cdot p \vee r$, because it can be shown to be a logical consequence of the remaining four. Now, a peculiarity of the way in which Hilbert and Ackermann deal with propositions [a]–[d] is this: they endeavor to show (pp. 29 ff.) that these propositions are, at once, *consistent*, *independent*, and *complete*, in the technical senses which these terms bear in connection with ordinary deductive systems. Their arguments in each of these cases call for comment, and we shall consider them in order.

In dealing with the question of consistency, the authors use an interpretational method, involving arithmetical products of 0 and 1. Of course, expressions [a]–[d] express propositions, and thus do not admit of interpretation as they stand; so that we must first abstract from the particular meanings of the symbols in question, and then re-interpret these symbols arithmetically. We need not concern ourselves here with the details of this argument; it involves showing that expressions [a]–[d], when given the arithmetical interpretation in question, are such that they, together with all the arithmetical products that can be derived from them by means of the two rules of deduction, have the value 0; a one-to-one correspondence is assumed to hold between these arithmetical propositions and the symbols, and again between the symbols and the original logical propositions; we are then told that if the original propositions could lead to a contradiction, some arithmetical product derivable from the primitive expressions would have to have the value 1.

It is somewhat surprising that the authors adopted such a roundabout procedure in endeavoring to establish a point that is immediately obvious; and the fact that they did do this leads us to suspect that the usual argument for consistency is not entirely clear to them. This usual argument, a form of which the authors adopt, is always one from *truth* to *consistency*; we know that no set of true propositions can be inconsistent, so that if we know that all the propositions of a set are true, we know that they cannot lead to contradiction. When we are concerned with a set of propositional functions, as in the ordinary case, we first assign values to the variables involved, and thus obtain a set of propositions, whose truth we may be able to observe; but, of course, when we are concerned from the outset with propositions, this first step, involving an interpretation, drops out. All we have to do, therefore, in order to see that propositions [a]–[d], together with the two rules, are consistent, is simply to observe that they are all plainly true. Hilbert and Ackermann seem to prefer to argue from the truth of certain arithmetical propositions to the consistency of the logical propositions with which they are concerned; but it is surely more to the point to argue directly from the obvious truth of these logical propositions.

In dealing with the question of independence, the authors employ interpretations similar to the one used in connection with consistency; they endeavor to show that no one of their primitive propositions follows from the remaining three. There is no occasion for examining this argument directly; it is quite certainly mistaken, for the sufficient reason that, in point of fact, [c] follows from [a], [b], and [d]. In order to see that [c] does follow, we may observe that a proposition of the form $p \vee q$ means simply that at least one of the two propositions in question is true, and that this is exactly what a proposition of the form $q \vee p$ means; so that these two symbols are not relevantly different. That is to say, the order in which p and q occur is not a symbolically significant feature of the symbol, just as the color of the ink is not, or the fact that the symbol is in italics. It is true that $q \vee p$ might, in some usage, be employed in a sense different from that assigned to $p \vee q$; but this cannot be so in connection with propositions, since only one meaning occurs, namely, "At least one of the two propositions is true"; there is simply nothing other than what is meant by $p \vee q$ left for $q \vee p$ to mean. But, now, since this is so, $p \vee q \cdot \supset \cdot q \vee p$ is merely a form of $p \supset p$; so that $(p, q) : p \vee q \cdot \supset \cdot q \vee p$ follows from $(p) \cdot p \supset p$; and this latter proposition is a consequence of [a], [b], and [d], as the authors show (p. 24).

With regard to completeness, two definitions are given, and an attempt is made to show that propositions [a]–[d] satisfy each of them. According to the first, a set of propositions is complete if, and only if, all true propositions of a certain domain can be derived from the set; and we are told that this means, in connection with the particular set in question, that all true universal propositions about functions of the sort occurring in the axioms can be derived. However, this restriction to universal propositions seems to be a mistake; for it seems clear that this definition ought to be understood to include true contradictories of universal propositions, and ought to mean that all these (that is, all that can be constructed in terms of the primitive notions employed) can be derived by means of the rules. If it does mean this, then the assumptions would seem to be not complete in this first sense; for such true propositions as $\sim(p, q)\cdot p \vee q$ can be formulated in terms of the primitive notions, and yet appear to be not derivable by the rules given. According to the second definition, a set is complete if, and only if, any proposition belonging to the domain in question is either implied by the primitive propositions or else incompatible with them. The authors establish the fact that this property holds by pointing out that all universal propositions of a certain kind follow from the axioms, and then showing that if other universal propositions followed, a contradiction would arise. Here, again, no explicit account is taken of contradictories of universal propositions; but it is easy enough to see that the argument applies to them. Thus, $(p, q)\cdot p \vee q$ is incompatible with the primitive propositions, so that $\sim(p, q)\cdot p \vee q$ does actually follow, despite the circumstance that the two rules seem to be not sufficient for effecting its deduction.

In order to be able to deal with propositions involving generalization with respect to variables denoting individuals, the authors add two further primitive propositions to their list (p. 53); namely, $(x)\cdot f(x)\cdot\supset\cdot f(y)$ and $f(y)\cdot\supset\cdot(\exists x)\cdot f(x)$, where $f(y)$ means, according to their usage, that f holds for any value of y. It may be said, with regard to the first of these propositions, that $f(y)$ means precisely what is meant by $(x)\cdot f(x)$, and that therefore the proposition here in question is an immediate consequence of $(p)\cdot p\supset p$, and is thus superfluous. There is really no significant difference in logic between "any value" and "every value," these alternative expressions being employed for psychological reasons merely; so that if we wish to use $f(y)$ in addition to $(x)\cdot f(x)$, we are free to introduce it through a purely verbal definition. All that is required, then, is the second proposition, which becomes $(x)\cdot f(x)\cdot\supset\cdot(\exists x)\cdot f(x)$. Here, however, a criticism of another sort is relevant: in contrast to the other assumptions, this second proposition does not express a purely logical truth. This is so because $(x)\cdot f(x)$, being universal, is negative, and would be true if there were not at least one value within the range of significance of x, whereas $(\exists x)\cdot f(x)$ would be false; so that although in point of fact the proposition in question is true, there is no reason in logic why it should be. It is really very doubtful whether propositions like this second one ought to be admitted into a purely logical analysis at all, except as subordinate constituents of more complex expressions.*

In their final chapter, the authors deal, as we have said, with certain paradoxes that appear when we attempt to apply generalization to functions. They discuss, in the first place, the paradox that arises in connection with the notion of a predicate as being predicable of itself. Let $f(t)$ mean "t is predicable of itself," so that $\sim f(t)$ will mean that t is not so predicable; and then consider the expressions $f(\sim f)$ and $\sim f(\sim f)$, with regard to which, it is easy to see that if they were significant, they would be logically equivalent, so that a logical vicious circle would arise. The authors consider, in the next place, the paradox that appears in connection with expressions like: "This proposition is false," or "I am now asserting a false proposition." Here two situations arise, which

* For a discussion of the relations of universal to particular propositions, see Mind, vol. 36 (1927), p. 342.

they do not distinguish. The first of these expressions is most naturally interpreted as a proposition, p, which asserts, "p is false"; so that we have a proposition directly about itself, in connection with which a genuine vicious circle arises. But the authors do not consider this case; instead, they take the most natural interpretation of the second of the foregoing expressions, according to which it is to be rendered: "There is one and only one proposition which I am asserting, and it is false." Clearly, this is often true; but it cannot be true if I assert it, nor can it be false; and yet a vicious circle does not arise, since all that follows is that I never do assert it. Nevertheless, the situation here in question is just as objectionable as the occurrence of a vicious circle, since it leads to the conclusion that a logically possible state of affairs is in fact impossible.

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NÖRLUND ON FINITE DIFFERENCES

Leçons sur les Équations Linéaires aux Différences Finies. By N. E. Nörlund. Paris, Gauthier-Villars, 1929. vi+153 pp.

The theory of finite difference equations arose in investigations by Lagrange and Laplace; but it is only recently that the properties of the solutions of these equations have been developed with any considerable detail. The modern researches on this subject have been inaugurated by Poincaré and Pincherle. One owes to Poincaré a remarkable theorem on the manner in which the solutions of a linear homogeneous equation in finite differences behave for very large values of the variable. This theorem has been the point of departure of several investigations. In his preface the author says: "In recent years the theory of finite difference equations has been developed by a large number of authors, among whom may be mentioned G. D. Birkhoff, H. Galbrun, E. Hilb, E. Bortolotti, O. Perron, R. D. Carmichael, J. Horn, K. P. Williams, A. Guldberg and G. Wallenberg. [To this list, of course, should be added the name of Nörlund himself.] The subject is too vast for it to be possible to give here an exposition of all the results obtained. The aim of this book is to put in evidence the essential properties of the solutions of linear homogeneous equations, on the one hand by aid of their development in factorial series, on the other hand by aid of certain methods of successive approximations due to G. D. Birkhoff and R. D. Carmichael." Chapters I–IV are devoted to a single linear equation, different hypotheses relative to the coefficients being made in the different chapters. Chapters V–VI treat similar problems for a system of linear equations.

The first chapter is devoted to general properties of linear equations, such as the existence theorems which are readily proved, adjoint equations, equations with second member, and the reduction of the order of an equation by means of known solutions. A large part of this chapter is elementary and is devoted to the preliminaries of the general theory. There is, however, an existence theorem of considerable importance, based on hypotheses of a broad general character. In the fifth chapter one finds a treatment of precisely similar ques-