BIRKHOFF ON DYNAMICAL SYSTEMS

Dynamical Systems. By G. D. Birkhoff. (American Mathematical Society Colloquium Publications, vol. IX.) 295 pp.

In these colloquium lectures the author presents in one volume the ideas and results which he has derived in the last score of years of his work on dynamical theory. It is *modern dynamics* in a sense to which the world has grown accustomed since the thought of Poincaré has become its intellectual heritage. The scope of the work is realized when it is considered in detail.

In Chapter I, the attempt is made to discover what physical meaning there may be in the assumption that the equations of dynamics are Lagrangian. The classical existence and uniqueness theorems for ordinary differential equations are first established. Next, starting with a general system of equations, the author makes physical assumptions of a general nature, with the object of discovering their analytical counterpart. Thus, a conservative system is defined (as in thermodynamics) as one in which the external forces accomplish zero work when the system describes a closed cycle, and from this property a restricted form is obtained from the equations of motion, which includes the Lagrangian as a special case.

Chapter II deals with the variational principles of dynamics. These are derived as usual for the classical systems, as well as for the somewhat more general ones considered in the first chapter. This is also done for Pfaffian systems, a sort of generalization of Hamiltonian systems which the author obtains by replacing in Hamilton's principle

$$\delta\int_{t_0}^{t_1}\left[\sum_{j=1}^{n} p_j q_j' - H\right] dt = 0,$$

by

$$\delta\int_{t_0}^{t_1}\left[\sum_{j=1}^{n} P_j p_j' + Q\right] dt = 0,$$

(P_j, Q being arbitrary functions of $p_1, \cdots, p_n$; n, even). The transformation theory, and such methods as the ignoration of coordinates, reduction by special integrals, etc., are developed for these systems. The Pfaffian systems have the advantage that they maintain their form under the general point transformation of the coordinates and momenta.

In the neighborhood of a regular point, all differential systems are equivalent under the group of analytic point transformations; they are devoid of invariantive characteristics. The simplest cases in which such characteristics can arise are for the neighborhood of a point of equilibrium, and for the neighborhood of a periodic solution, which can be reduced to a generalized point of equilibrium. Our attention is thus naturally directed to such motions, which are, moreover, of considerable physical importance. In Chapter III, these motions are considered, and a complete treatment of those of their invariantive properties which are of a purely formal nature is given in the general case (where there are no linear relations with integral coefficients between the multipliers). All questions of convergence are systematically laid aside as being irrelevant to the matter in hand. Functions are regarded merely as formal power series in the variables (the coefficients being periodic in the time in the case of generalized equilibrium), and the transformations and other operations, as entirely formal processes. The reduction of the differential system to standard forms is accomplished with the aid of the group of formal transformations. Hamiltonian and Pfaffian systems are then considered in detail. Nothing is said about asymptotic or other possible relations between the series and the functions in the case where the former fail to converge. By this sharp separation from the rest, the formal properties are brought into a clearer light than in the classical treatment of this subject, where they are never given true recognition and play but a subsidiary part.

Chapter IV continues the treatment of the invariantive characteristics in the neighborhood of equilibrium or periodic motion, occupying itself with the formal stability of such motions. A result of the classical theory is that when the multipliers are pure imaginaries there is "stability of the first order," that is, approximate stability, which becomes exact when all but terms of the first order in the distance from the periodic motion are omitted from the differential equations. The author defines an extension, due in the main to Poincaré, of this concept to terms of all orders: "complete formal or trigonometric stability." This must not be confused with actual stability in the physical sense, the discovery of whose relation with the former is an outstanding problem in the theory. It is shown that in the case of Hamiltonian and Pfaffian systems stability of the first order implies complete formal stability. Conversely, if there is such stability, the system can be taken into these canonical forms near the motion considered, by means of the formal group. This is a contribution to the problem of Chapter I,—the meaning of the canonical form. Finally, reversibility and its connection with stability are treated.

Chapter V deals with the various general methods by means of which the existence of periodic motions may be established; in this sense it supplies a foundation for the two preceding chapters. The three known types of method are described, with examples. The first consists of those (due to Hadamard, Whittaker, Hilbert, Signorini, and Birkhoff) which make use of the variational principles of dynamics; they are best illustrated by considering the important special case of geodesics on a surface. Suppose, first, that a smooth elastic string in the form of a loop be slipped onto a smooth surface possessed of a constriction, and remain binding the constriction, in a stretched condition: it will form a closed geodesic on the surface. This idea and its extension to broad classes of dynamical systems establishes the existence of periodic motions of the "minimum" type. Suppose, secondly, that it is attempted to pass a convex surface through an elastic loop which is too small for it to go through without stretching. Consider the totality of modes of passing the surface through the loop: there will be a lower limit to the maximum stretching required, and in the corresponding mode of passage the elastic will, in its position of greatest stretching, form a closed geodesic. This method, greatly generalized with the aid of geodesic polygons, etc., establishes the existence of periodic motions of the "minimax" type. The second type of method consists in the analytic continuation of a periodic orbit, known to exist for a particular value of a parameter, to neighboring values. This method (due to Hill and Poincaré) is briefly outlined. The third type of method consists in reducing the dynamical problem to a point transformation, the periodic motions appearing as fixed points under one of its powers. It is merely introduced in this chapter, and illustrated by an example having three degrees of freedom.

Chapter VI is entitled Poincaré's Geometric Theorem. The third type of method of Chapter V is studied in detail in the only case where it has been applied with any generality, namely, that of a conservative system with two degrees of freedom. In such a system the states of motion (position and velocity) constitute a 4-dimensional manifold, in which those states corresponding to a given energy constant form a 3-dimensional sub-manifold. The equations of motion define a vector field in the latter. When this is interpreted as a velocity field, it appears that the motions of the system are coextensive with a permanent flow of a fluid situated in the manifold of states of motion. The integral invariant of the system has its counterpart in the incompressibility of the flow. In the case where there is a uniform swirling tendency in the flow, a surface may be interposed which is cut by every stream line, in the same sense, and at least once in any interval of time less than a definite constant independent of the stream-line. Such a surface is known as the *surface of section*. The stream-lines, by their successive intersections, define a point transformation of the surface of section into itself, which is essentially area-preserving, on account of incompressibility of the flow. Periodic motions correspond to closed stream-lines, and these, in turn, to fixed points under a power of the transformation. Their properties of stability, etc., are reflected in properties of the transformation. In this sense the dynamical problem is reduced to the study of a surface transformation. It is, of course, to be noted that there are important properties which are not represented by the transformation, and hence the reduction of the problem is by no means a complete one. In the present chapter the very difficult problem of establishing the existence of the surface of section is treated for special classes of systems. At this point the author makes what must be regarded as a notable advance in the theory, by establishing a theorem which is the analog *im Kleinen* of Poincaré's geometric theorem, an extension the value of which lies in the fact that it can be applied to the case where the existence of only a local surface of section is known, —which is the general situation. Finally, by considering in detail the reflections of a billiard ball rolling on a flat table with a curved boundary (the simplest non-integrable case), the author brings into a clear light every point of the theory; we regard this as a masterpiece of presentation.

Chapter VII opens with the statement:

"The final aim of the theory of the motions of a dynamical system must be directed toward the qualitative determination of all possible types of motions and of the interrelation of these motions.

"The present chapter represents an attempt to formulate a theory of this kind."

The author defines "wandering" and "non-wandering" motions, "central" motions, "recurrence" and "transitivity" of motions, and establishes a series of theorems concerning their existence and interrelations. The only property made use of in this chapter is the bare fact that the curves are integral curves of analytic differential equations. The treatment has the aspect of a study in point-set theory.

Chapter VIII continues with the work of Chapter VII, in the case of a dynamical system with two degrees of freedom, but is far more complete in the properties which it obtains, and in the analytical form in which the results are frequently given. This is made possible by the formal developments and the theory of surface transformations of Chapters IV and VI. The conceptions of this chapter are the logical descendents of Poincaré's *Théorie des Conséquents*, just as those of the preceding chapter are of the intellectual lineage of that great author's thoughts on the stability of Poisson. We have in the present chapter a synthesis of two disciplines: the extreme of formalism on the one hand, and the pure theory of point sets on the other; and it is this union of ideas which gives so rich a yield not only of theorems which this chapter formulates, but of those theorems which are unwritten and which rise in the mind of the reader.

Chapter IX, the final chapter of the work, is devoted to the problem of three bodies, particularly from the point of view of Sundman's researches. On the basis of certain inequalities taking place among certain of the variables in the differential equations and their derivatives, Sundman has shown that except when the vector angular momentum is zero, triple collision is impossible. Other forms of irregularity being regularizable, it follows that series exist in the general case, convergent for all values of the time, which give the motion in explicit form; this is the famous "solution" of Sundman of the problem of three bodies. These results, presented in an original and clear form, constitute the bulk of the present chapter. Towards the end, the author returns to some of the ideas developed in earlier chapters and applies them with the aid of Sundman's work to the problem of three bodies.

When we turn from considering the work in its detailed aspects to regarding it as a whole, our task is difficult: we attempt to review a book, and find that we are commenting on a theory. This very circumstance is more eloquent than any words of our own.

How is the value of a mathematical discipline to be judged? Is it in the view which it discloses of the physical world, and which could not be had without its aid? If this is the case, then the developments which we are considering promise much, since they are, in their historical origin, purely physical. But they have traveled a great distance from their beginnings, and have taken the aspect of a deep analysis of methods which have proved effective in the past, rather than that of the direct study of physical reality. Moreover, there is one circumstance which is ominous, and that is that the properties most often considered are properties which are changed altogether by an infinitely small change in the physical conditions attendant on the problem, or by the slightest change in initial data. In the world of measurement such properties find little place.

Is the value of a discipline to depend on its relations with other parts of the science? Our review has been unsuccessful indeed if it has failed to bring into light not only the broad connections of the subject with the existing science, but the promise which it gives of initiating further developments in mathematics.

But there is an intellectual criterion which is ultimate, which differs from those others which seek the value of a theory outside the theory itself. It is the judgment of its esthetic worth, the appreciation of its structure. It is perhaps more noteworthy that dynamical theory should have acquired a value of this sort than, for example, projective geometry, or the theory of groups. And to our author more perhaps than to any other man belongs the credit that this is so.

B. O. KOOPMAN

THE RHIND PAPYRUS

The Rhind Mathematical Papyrus, British Museum 10057 and 10058, in two volumes. Volume I. By Arnold Buffum Chace, with the assistance of Henry Parker Manning, and with a bibliography of Egyptian mathematics by Raymond Clare Archibald. [x]+210 pp., 1927. Royal 8vo. Volume II. By Arnold Buffum Chace, Ludlow Bull, and Henry Parker Manning, with a bibliography of Egyptian and Babylonian mathematics (supplement) by Raymond Clare Archibald, and a description of the mathematical leather roll in the British Museum, by S. R. K. Glanville. Mathematical Association of America, Oberlin, Ohio; xvi pp.+31 photographic plates +109 facsimile plates+109 facing pages of text, 12 pp. of bibliography +8 pp., 1929. Royal oblong folio. Price, $20.

The publication of this treatise, the product of nearly twenty years of scholarly work, is an event of such importance in connection with the history of mathematics as to require more than a cursory examination or a brief description. The Rhind (Ahmes, A'h-mosè) Papyrus is the most extensive mathematical treatise written before the sixteenth century B. C. that has come down to us. We have no contemporary manuscripts of any of the Greek classics on geometry, the theory of numbers, or computation. Our knowledge of the Sumerian, Assyrian, Babylonian, and Chaldean mathematics is derived solely from numerical tables, a few tablets containing a little work in mensuration, numerous others relating to commercial life, and some recently studied ones relating to the Pythagorean triangle, the angle inscribed in a semi-circle, and the rule for solving the quadratic. Such Chinese and Hindu sources as we have, relating to the pre-Christian period, are of uncertain authenticity, especially those purporting to be copies of Chinese documents preceding the eleventh century B. C. In the case of Egypt, however, we have, in fairly complete form, the original document written by A'h-mosè (Ahmes) in the reign of ʻA-user-Rê' (c. 1650 B. C.), being a copy or a paraphrase of one dating from the reign of Ne-maʻet-Rê' (Amen-em-hât III), 1849–1801 B. C., or at least similar to it. That such a document, written more than a thousand years before mathematics began to make any noteworthy advance in Greek territory, should have come down to us almost intact, is one of the most remarkable incidents connected with source material of any kind. It is also interesting to know that another manuscript, even earlier than this, is soon to be published, the Golenishchev papyrus now in Moscow (*Quellen und Studien zur Geschichte der Mathematik, Abteilung A*: *Quellen*, Berlin, 1930),