HAUSDORFF'S THEOREM CONCERNING HERMITIAN FORMS

BY M. H. STONE

In 1919, Hausdorff proved an interesting theorem to the effect that the complex values assumed by the Hermitian form

$$\sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}\bar{x}_\alpha x_\beta,$$

when the complex numbers $x_1, \cdots, x_n$ take on all values compatible with the relation

$$\sum_{\alpha=1}^{n} \bar{x}_\alpha x_\alpha = 1,$$

constitute a convex point set in the complex plane.* In his demonstration Hausdorff employs the transformation of Hermitian symmetric forms of this type to principal axes. It is obvious, therefore, that the extension of this method to more general instances is impossible, because the principal axis transformation may no longer be available, a situation which arises, for instance, in the important case of unbounded forms in infinitely many variables. On this account, a proof along less special lines is of considerable interest. It is gratifying to ascertain that an entirely elementary, explicit, and general demonstration of Hausdorff's theorem can be devised by making it depend upon the theorem for binary forms. The proof we shall give involves nothing more difficult than manipulations of complex numbers and the solution of quadratic equations.

We first indicate what we shall mean, abstractly, by an Hermitian bilinear form. Let $\mathfrak{J}$ be a class of elements in which the operations $+$ (vector addition) and $a\cdot$ (scalar multiplication by an arbitrary complex constant a) are significant and possess their customary algebraic properties; let $\mathfrak{J}$ be closed under these operations. Such a class may be called a complex vector space, its elements complex vectors. A complex-valued

* Mathematische Zeitschrift, vol. 3 (1919), pp. 314–316.

function $B(f, g)$ defined for every pair of elements (f, g) in $\mathfrak{J}$ is said to be an Hermitian bilinear form if and only if it has the following properties:

$$\begin{aligned} B(af,g) &= aB(f,g), \\ B(f_1+f_2,g) &= B(f_1,g)+B(f_2,g), \\ B(f,ag) &= \bar{a}B(f,g), \\ B(f,g_1+g_2) &= B(f,g_1)+B(f,g_2). \end{aligned}$$

Such a form is said to be Hermitian symmetric and positive definite if and only if $B(f, g)$ and $B(g, f)$ are conjugate complex numbers and the real number $B(f, f)$ is never negative, vanishing only when f is the null element or null vector. When B is Hermitian symmetric and positive definite, we can establish the Cauchy-Schwarz inequality

$$|B(f,g)| \leqq [B(f,f)B(g,g)]^{1/2},$$

by the usual considerations.

We assume that there exist in $\mathfrak{J}$ two Hermitian bilinear forms B_1 and B_2, the second of which is Hermitian symmetric and positive definite. Let W be the set of complex values assumed by B_1 when B_2 is required to have the value 1. We wish to show that W is a convex set; that is, that, whenever z_1 and z_2 are in W, the points of the line segment joining them are also in W.

If W contains just one point, there is nothing to prove, so that this case may be ruled out. Now, let z_1 and z_2 be distinct complex numbers in W, and f_1 and f_2 elements or vectors in $\mathfrak{J}$ such that

$$B_1(f_1,f_1) = z_1, \quad B_1(f_2,f_2) = z_2, \quad B_2(f_1,f_1) = 1, \quad B_2(f_2,f_2) = 1.$$

We consider the binary Hermitian forms

$$\begin{aligned} A_1(x_1,x_2) &= B_1(x_1f_1+x_2f_2, x_1f_1+x_2f_2) \\ &= z_1\bar{x}_1x_1 + B_1(f_2,f_1)\bar{x}_1x_2 + B_1(f_1,f_2)x_1\bar{x}_2 + z_2x_2\bar{x}_2, \\ A_2(x_1,x_2) &= B_2(x_1f_1+x_2f_2, x_1f_1+x_2f_2) \\ &= \bar{x}_1x_1 + B_2(f_2,f_1)\bar{x}_1x_2 + B_2(f_1,f_2)x_1\bar{x}_2 + x_2\bar{x}_2. \end{aligned}$$

We must show that, when A_2 is restricted to have the value 1,

A_1 takes on all values represented by points of the line segment joining z_1 and z_2.

It is convenient to introduce a new binary form

$$A(x_1, x_2) = (A_1 - z_2A_2)/(z_1 - z_2) = \bar{x}_1x_1 + a_{12}\bar{x}_1x_2 + a_{21}x_1\bar{x}_2.$$

In order that A_1 should exhibit the requisite behavior, it is necessary and sufficient that, while $A_2 = 1$, A should take on all real values from 0 to 1 inclusive. We shall now find values for x_1 and x_2 which bring about the desired result. We first define a complex number γ as follows: $\gamma = \pm 1$ when $\bar{a}_{12} = a_{21}$, and $\gamma = \pm(\bar{a}_{12} - a_{21})/|\bar{a}_{12} - a_{21}|$ when $\bar{a}_{12} \neq a_{21}$, the plus or the minus sign being chosen in each case so as to render the real number $\beta = \Re(\gamma B_2(f_2, f_1))$ not negative. We set $x_1 = x$, $x_2 = \gamma y$, where x and y are real variables, and substitute these values in the expressions for A and A_2, obtaining

$$\begin{aligned} A &= x^2 + \alpha xy, \\ \alpha &= \bar{\alpha} = \pm(a_{12} + a_{21}) \quad \text{or} \quad \pm(a_{12}\bar{a}_{12} - a_{21}\bar{a}_{21})/|\bar{a}_{12} - a_{21}|, \\ A_2 &= x^2 + 2\beta xy + y^2. \end{aligned}$$

We observe that $0 \leqq \beta \leqq 1$, in view of the inequalities

$$\begin{aligned} |\beta| = |R(\gamma B_2(f_2, f_1))| &\leqq |\gamma||B_2(f_2, f_1)| \\ &= |B_2(f_2, f_1)| \leqq [B_2(f_1, f_1)B_2(f_2, f_2)]^{1/2} = 1. \end{aligned}$$

Hence the equation $A_2 = 1$ can be satisfied by taking $y = -\beta x + [1 - (1 - \beta^2)x^2]^{1/2}$, and, with this value for y, the form A becomes

$$A = (1 - \alpha\beta)x^2 + \alpha x[1 - (1 - \beta^2)x^2]^{1/2}.$$

For $x = 0$, A takes on the value 0, for $x = 1$ the value 1; since A is a continuous real-valued function of x, it takes on at least once every real value from zero to one when x varies between these same values, as we wished to show.

This completes the demonstration of Hausdorff's theorem that W is a convex set.

HARVARD UNIVERSITY