NOTES ON THE RATIONAL PLANE OSCNODAL QUARTIC CURVE*

BY J. H. NEELLEY

1. *Introduction.* In a recent paper†, which discussed the covariants of the curve R_2^4 with an oscnode, I made the statement that the quartic $\overline{12}^2(1, 2)$, where 1 and 2 refer to the base quartics $(\beta t)^4$ and $(\gamma t)^4$ of the fundamental involution, seemed to represent four collinear points of the curve when and only when the three nodes were adjacent in such a manner as to form an oscnode. In arriving at this conclusion it happened that, for each type of the curve examined, I chose as base quartics of the fundamental involution at least one catalectic quartic.‡ This accidental choice lead to the statement mentioned above. The failure of a later attempt to make the regular oscnodal invariants vanish under the condition that $\overline{12}^2(1, 2)$ should represent a line section of the curve caused the present investigation. In this paper, I shall discuss the form $\overline{12}^2(1, 2)$ and associated covariants and I shall develop several theorems which give new necessary and sufficient conditions for the oscnodal singularity.

2. *Certain Conditions.* I shall refer the curve to the tangents at the two points $t=0$ and $t=\infty$ and the line joining those points. This reference scheme assumes evidently that $t=0$ and $t=\infty$ are merely any two points of the curve such that the line of the curve at either does not pass through the other point. So this general reference triangle gives the curve as follows:

$$(1)\qquad \begin{cases} x_0 = a_0t^4 + b_0t^3 + c_0t^2, \\ x_1 = \qquad\quad b_1t^3 + c_1t^2 + d_1t, \\ x_2 = \qquad\qquad\quad c_2t^2 + d_2t + e_2, \end{cases}$$

where neither a_0 nor e_2 can be zero, and in order to avoid some

* Presented to the Society, December 27, 1929.

† Neelley, *Effects of the oscnode upon covariant forms of the rational plane quartic curve*, this Bulletin, vol. 35, pp. 571–575.

‡ Neelley, *Compound singularities of the rational plane quartic curve*, American Journal of Mathematics, vol. 49 (1927), pp. 389–400.

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singularity at $t=0$ or $t=\infty$ we shall say that neither b_1 nor d_1 shall vanish. This avoids at least a cusp at either point. Then we have

$$(2)\quad \begin{cases} (\beta t)^4 \equiv a_0 b_1 d_2 t^4 + 4a_0 b_1 e_2 t^3 - 4a_0 d_1 e_2 t - b_0 d e_2, \\ (\gamma t)^4 \equiv a_0(c_1 d_2 - c_2 d_1)t^4 + 4a_0 c_1 e_2 t^3 + 6a_0 d_1 e_2 t^2 - c_0 d_1 e_2; \end{cases}$$

whence the form $\overline{12}^2(1, 2)$ except for the factor $a_0 e_2$ is

$$(3)\quad \begin{cases} (a_0 b_1 d_1 d_2 - 2a_0 b_1 c_1 e_2)t^4 + 2(a_0 c_2 d_1^2 - a_0 c_1 d_1 d_2 - a_0 b_1 d_1 e_2)t^3 \\ \quad + (b_0 c_2 d_1^2 - b_1 c_0 d_1 d_2 - b_0 c_1 d_1 d_2 - 2a_0 c_1 d_1 e_2)t^2 + 2(a_0 d_1^2 e_2 \\ \quad - b_1 c_0 d_1 e_2 - b_0 c_1 d_1 e_2)t - b_0 d_1^2 e_2. \end{cases}$$

This form and expressions (2) give as the conditions that $\overline{12}^2(1, 2)=0$ be a line section

$$(4)\quad B \equiv 2b_0 b_1 c_1 e_2 - b_0 b_1 d_1 d_2 - 2a_0 b_1 d_1 e_2 - a_0 c_1 d_1 d_2 + a_0 c_2 d_1^2 + b_1^2 c_0 e_2 = 0,$$

$$(5)\quad C \equiv 2b_1 c_0 c_1 e_2 + b_0 c_1^2 e_2 - 2a_0 c_1 d_1 e_2 - b_1 c_0 d_1 d_2 - b_0 c_1 d_1 d_2 + b_0 c_2 d_1^2 = 0,$$

where we have not written the non-vanishing factor $2a_0 d_1 e_2$ for each form.

Recalling that the catalectic sets of the fundamental pencil are associated with the nodes in such a way that each set fixes a node, let us consider the catalecticant of the form $\lambda(\beta t)^4+\mu(\gamma t)^4$. Except for the factor $a_0^2 d_1 e_2^2$, it may be written in the form

$$(6)\qquad A\lambda^3 + B\lambda^2\mu + C\lambda\mu^2 + D\mu^3 = 0,$$

where $A \equiv b_0 b_1^2 e_2 - a_0 b_1 d_1 d_2$, B and C are forms (4) and (5) respectively, and $D \equiv c_0 c_1^2 e_2 - a_0 d_1^2 e_2 - c_0 c_1 d_1 d_2 + c_0 c_2 d_1^2$. Consideration of conditions (4) and (5) shows that this catalecticant becomes

$$(7)\qquad A\lambda^3 + D\mu^3 = 0$$

when $\overline{12}^2(1, 2)=0$ is a line section. This gives the theorem:

If the form $\overline{12}^2(1, 2)$ of the fundamental pencil of the curve R_2^4 represents a line section, the catalecticant of the fundamental quartics reduces to the sum of two cubes.

The cubic (6) shows that the conditions (4) and (5) are not sufficient conditions to make nodes of the curve coincident.

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But forms A and D are the conditions that the quartics (2) be catalectic respectively. So, if we pick as either $(\beta t)^4$ or $(\gamma t)^4$ one of the catalectic quartics of the fundamental pencil, the cubic (6) reduces to a perfect cube when $\overline{12}^2(1, 2)=0$ is a line section. Before stating the theorem this seems to give, let us consider the invariants which vanish when the curve has an oscnode.* With no loss of generality we assume $c_1=d_2=0$ in the representation (1) of the curve. Then the forms considered become

$$(8)\qquad \begin{cases} A \equiv b_0 b_1^2 e_2, & B \equiv a_0 c_2 d_1^2 - 2a_0 b_1 d_1 e_2 + b_1^2 c_0 e_2, \\ C \equiv b_0 c_2 d_1^2, & D \equiv c_0 c_2 d_1^2 - a_0 d_1^2 e_2, \end{cases}$$

except for factors involving a_0, d_1, and e_2. Obviously the tacnode invariant I_6' vanishes as the cubic has a triple root. So we need only observe the additional oscnodal invariant $\bar{I}_4$. This becomes

$$(9)\qquad \begin{cases} 64a_0^3 d_1^3 e_2^3 \left[2A(27a_0 b_0 c_2 d_1^2 e_2 - 2b_0 c_0 c_2^2 d_1^2) + 2B(14a_0 b_1 c_0 c_2 d_1 e_2\right. \\ \quad - 3a_0^2 c_2 d_1^2 e_2 - 6a_0^2 b_1 d_1 e_2^2 - 3a_0 b_1^2 c_0 e_2^2 - a_0 c_0 c_2^2 d_1^2 \\ \quad \left. - b_1^2 c_0^2 c_2 e_2) + C(a_0 b_0 c_2^2 d_1^2 - 10a_0 b_0 b_1 c_2 d_1 e_2)\right]. \end{cases}$$

Therefore $A=B=C=0$ are sufficient conditions for $\bar{I}_4$ to vanish. Relations (8) show that A and C are not independent and $B=C=0$ are sufficient to make $\bar{I}_4=0$ but they do not make $I_6'=0$. However, the independence of A and C is assured if we take only $c_1=0$ in the equations (1). In this case too we find that $A=B=C=0$ are sufficient to make $\bar{I}_4$ vanish. We may now state these results as the following theorem.

THEOREM 1. *If the fundamental pencil of quartics of the curve be so chosen that one of the base quartics is catalectic, then the necessary and sufficient condition that the curve have an oscnode is that the covariant* $\overline{12}^2(1, 2)$ *represent a line section of the curve.*

The appearance of the forms B and C in the catalecticant naturally created a suspicion that they might occur in some of the forms of the system of the binary quartics (2). And it appears that if we operate with $(\beta t)^4$ upon the Hessian of $(\gamma t)^4$ and vice versa, the results are respectively

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* Neelley, loc. cit., American Journal of Mathematics, vol. 49, p. 398.

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(10) $\overline{12}^4(1,H_2) \equiv a_0^2 d_1 e_2^2 \cdot C, \qquad \overline{12}^4(2,H_1) \equiv a_0^2 d_1 e_2^2 \cdot B.$

These allow us to write the catalecticant cubic (6) in the following manner:

$$g_{3(1)}\lambda^3 + \overline{12}^4(2,H_1)\lambda^2\mu + \overline{12}^4(1,H_2)\lambda\mu^2 + g_{3(2)}\mu^3 = 0, \tag{11}$$

showing that its coefficients are these invariants of the two quartics $(\beta t)^4$ and $(\gamma t)^4$. The forms (10) also give the following theorem.

THEOREM 2. *The form* $\overline{12}^2(1, 2)$ *is a line section of the curve when each base quartic of the fundamental pencil is apolar to the Hessian of the other. And when the proper choice of* $(\beta t)^4$ *or* $(\gamma t)^4$ *is made the curve has an oscnode if the base quartics are each apolar to the Hessian of the other and conversely.*

In testing this last theorem for the types of the curve with an oscnode we find that having picked for $(\beta t)^4$ a catalectic quartic the choice of $(\gamma t)^4$ is unrestricted and still each quartic is apolar to the Hessian of the other. But this is not true for any two quartics of the pencil. Then, on the assumption that $(\beta t)^4$ is catalectic, we find that any other member of the pencil such as $\lambda_1(\beta t)^4+\mu_1(\gamma t)^4$ is apolar to the Hessian of $(\beta t)^4$ and conversely if $\overline{12}^2(1, 2)$ represents a line section. The converse is readily shown when the general representation (1) of the curve is used.

In a previous article* we made the statement that the oscnodal cusp requires that each of the self-apolar members of the fundamental pencil be apolar to the Hessian of the other in addition to the invariant conditions. We find here that this is not the whole truth. For we have shown that this is true for the base quartics, if one is catalectic, for any type of the curve with an oscnode. This apparent inconsistency is at once removed, however, when we observe that if the oscnode is a cusp or a knot one of the self-apolar quartics is also catalectic. The situation is as follows.

THEOREM 3. *The curve with an oscnodal cusp has two distinct self-apolar members in its fundamental pencil, one of which is*

* Neelley, loc. cit., American Journal of Mathematics, vol. 49 (1927), p. 398.

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catalectic, while the knot is such that the self-apolar members coincide. The knot then has one form representing both of its possible self-apolar forms and also its three coincident catalectic forms.

This shows that the previous statement is true but that it is a special case of the general situation, derived in this paper, for all types of the curve with an oscnode.

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THE TRANSFORMATIONS GENERATED BY AN INFINITESIMAL PROJECTIVE TRANSFORMATION IN FUNCTION SPACE*

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The problem considered in this note is to determine the one-parameter group of finite transformations generated by an infinitesimal projective transformation in the space of continuous functions. It will be shown that this one-parameter group consists entirely of projective transformations of function-space in the sense defined by L. L. Dines.† While the problem is completely solved in the paper just cited (p. 57), it will be seen that the result may be obtained directly without the use of the auxiliary formulas developed by Dines in the first part of his paper.

Let the given infinitesimal projective transformation be defined by the integro-differential equation

$$(1)\qquad \frac{\partial\phi'(x;t)}{\partial t} = \lambda(x) + \mu(x)\phi'(x;t) + \int_0^1 \nu(x,y)\phi'(y;t)dy - \phi'(x;t)\int_0^1 \rho(y)\phi'(y;t)dy$$

with the boundary condition $\phi'(x;0)=\phi(x)$. It is required to

* Presented to the Society, April 6, 1928.

† *Projective transformations in function space*, Transactions of this Society, vol. 20 (1914), p. 45.

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