SOME PROPERTIES OF SPHERICAL HARMONICS*

BY H. BATEMAN

A Newtonian potential $V(x, y, z)$ can often be derived from a four-dimensional potential $W(x, y, z, w)$ by forming the definite integral

$$V = \frac{1}{\pi}\int_{-\infty}^{\infty} W dw.$$

Thus, if W is the reciprocal of $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2+(w-w_0)^2$, where x_0, y_0, z_0, w_0 are real constants, the Newtonian potential V is the inverse of the distance between the points (x, y, z) and (x_0, y_0, z_0). There is thus a simple correspondence between the *charges* giving rise to the two potentials, the point charge in the three-dimensional space, S_3, being simply the projection of the corresponding point charge in the four-dimensional space S_4. With suitable restrictions this method of projection may be applied to surface distributions of charge in S_4 and we shall consider in particular the case of a continuous distribution over the spherical surface

$$x^2 + y^2 + w^2 = a^2, \quad z = 0,$$

when the surface density depends only on x^2+y^2. In this case

$$W = \int_0^{\pi}\int_0^{2\pi} (a^2/R^2) f(\cos\theta) \sin\theta d\theta d\phi ,$$

where $f(\cos\theta)$ is the function giving the law of density and

$$R^2 = (w - a\cos\theta)^2 + z^2 + (x - a\sin\theta\cos\phi)^2 + (y - a\sin\theta\sin\phi)^2.$$

To reduce the integral to a simpler form we write

$$\cos\theta = \zeta, \qquad x^2 + y^2 = \rho^2, \qquad \rho^2 + w^2 = \tau^2;$$

then

$$W = (\pi a/\tau) U(X, Y, Z),$$

where

* Presented to the Society, June 20, 1929.

$$U = \int_{-1}^{1} [(\zeta - Z)^2 + X^2 + Y^2]^{-1/2} f(\zeta) d\zeta,$$

$$Z = ws^2/(2a\tau^2), \quad X^2 + Y^2 + Z^2 = (s^4 - 4a^2\rho^2)/(4a^2\tau^2),$$

$$s^2 = x^2 + y^2 + z^2 + w^2 + a^2.$$

Now, when (X, Y, Z) are regarded as rectangular coordinates in a new space S_3^*, the function U is the Newtonian potential of a rod of line density $f(\zeta)$ and so it is advantageous to introduce spheroidal coordinates ξ, η, of the prolate type, by means of the equations

$$\cos \xi \cosh \eta = Z = ws^2/(2a\tau^2),$$

$$\sin \xi \sinh \eta = (X^2 + Y^2)^{1/2} = \rho(s^4 - 4a^2\tau^2)^{1/2}/(2a\tau^2).$$

We then find that

$$\cos \xi = w/\tau, \qquad \cosh \eta = s^2/(2a\tau).$$

If $P_n(\mu)$, $Q_n(\sigma)$ are the usual Legendre functions, the potential

$$U = Q_n(\cosh \eta) P_n(\cos \xi)$$

gives rise to the four-dimensional potential

$$W = \pi(a/\tau) Q_n(s^2/(2a\tau)) P_n(w/\tau).$$

This is a particular case of the theorem that if $H(x, y, w)$ is a homogeneous function of x, y, w of degree $-n-1$ such that

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{\partial^2 H}{\partial w^2} = 0,$$

and T is a function of z and τ satisfying

$$(1) \qquad \frac{\partial^2 T}{\partial \tau^2} - \frac{2n}{\tau} \frac{\partial T}{\partial \tau} + \frac{\partial^2 T}{\partial z^2} = 0,$$

the product $W = HT$ is a four-dimensional potential. It should be noted that the equation (1) is satisfied by $\tau^n P_n(s^2/(2a\tau))$ and $\tau^n Q_n(s^2/(2a\tau))$. To ascertain the nature of the function V derived from W we note that when $\rho = 0$ the potential V becomes

$$a\int_{-\infty}^{\infty}\frac{dw}{w}Q_n\left(\frac{z^2+a^2+w^2}{2aw}\right)=\frac{a}{2}\int_{-\infty}^{\infty}dw\int_{-1}^{1}\frac{2aP_n(\mu)d\mu}{z^2+a^2+w^2-2a\mu w}$$
$$=a^2\int_{-1}^{1}d\mu\int_{-\infty}^{\infty}\frac{P_n(\mu)dw}{(w-a\mu)^2+a^2(1-\mu^2)+z^2}$$
$$=a^2\pi\int_{-1}^{1}[z^2+a^2(1-\mu^2)]^{-1/2}P_n(\mu)d\mu$$
$$=2a\pi C_m\cdot q_{2m}(|z|/a),\qquad (n=2m),$$
$$=0,\qquad (n=2m+1),$$

where

$$C_m=\frac{1\cdot 3\cdot\cdots\cdot(2m-1)}{2\cdot 4\cdot\cdots\cdot(2m)},$$

and where

$$q_{2m}(\sigma)=\frac{\pi^{1/2}\Gamma(2m+1)}{2^{2m+1}\Gamma(2m+3/2)}(1+\sigma^2)^{-m-1/2}\cdot F\left(m+\frac{1}{2},\ m+\frac{1}{2};2m+\frac{3}{2};\frac{1}{1+\sigma^2}\right)$$

is the type of Legendre function used for the standard harmonics associated with an oblate spheroid, the notation being that used in Lamb's *Hydrodynamics*, 5th edition, p. 124. The change of the order of integration in the repeated integral in the above analysis is easily justified when $z^2>\epsilon^2>0$ because Weierstrass' test may be used to establish the uniform convergence of the infinite integral in the range $-1\leqq\mu\leqq 1$.

The function V is the potential of a circular disc charged with a surface density $[a^2/(a^2-\rho^2)]^{1/2}P_{2m}[(1-\rho^2/a^2)^{1/2}]$, the charges on the two sides being equal. By introducing the spheroidal coordinates σ, μ, defined by the equations

$$z=a\mu\sigma,\quad \rho=a[(1-\mu^2)(1+\sigma^2)]^{1/2},$$

the potential V may be expanded in a unique manner in a series of spheroidal harmonics of type $A_nP_n(\mu)q_n(\sigma)$; and by putting $\mu=1$ it is easily seen that V is identical with the harmonic $2a\pi C_mP_{2m}(\mu)q_{2m}(\sigma)$. Hence

$$(2)\qquad \int_{-\infty}^{\infty}\frac{dw}{\tau}Q_{2m}(s^2/(2a\tau))P_{2m}(w/\tau)=2\pi C_mP_{2m}(\mu)q_{2m}(\sigma).$$

This is a relation between spheroidal harmonics of the prolate and oblate types.

On the disc we have $\sigma=0$, and since $2q_{2m}(0)=\pi C_m$, the value of V is

$$V_0 = \pi^2 a C_m{}^2 P_{2m}(\mu).$$

Another relation, which is obtained by the method of projection from S_4, may be mentioned here. If $r^2=x^2+y^2+z^2$, $\rho^2=x^2+y^2$,

$$(3)\quad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dw}{(r^2+w^2)^{n+1}}P_n\left(\frac{z^2+w^2-\rho^2}{z^2+w^2+\rho^2}\right) = C_n r^{-2n-1}P_{2n}\left(\frac{x}{r}\right).$$

If $w=r\tan\epsilon$, it is easily seen that the integral is of the form $r^{-2n-1}F(z/r)$, where $F(\nu)$ is a polynomial. Since the integral represents a Newtonian potential, $F(\nu)$ must be a constant multiple of $P_{2n}(\nu)$ and the constant multiplier may be identified with C_n by putting $\rho=0$.

The associated relation

$$(4)\quad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dw}{(r^2+w^2)^{n+1}}Q_n\left(\frac{z^2+w^2-\rho^2}{z^2+w^2+\rho^2}\right) = C_n r^{-2n-1}Q_{2n}\left(\frac{z}{r}\right)$$

is more difficult to prove. The integral may be shown by differentiation to be a Newtonian potential and the substitution $w=r\tan\epsilon$ indicates that it is of the form $r^{-2n-1}[A_nQ_{2n}(z/r)+B_nP_{2n}(z/r)]$, where A_n and B_n are constants to be determined. Now $Q_{2n}(0)=0$, $P_{2n}(0)=(-1)^nC_n$, hence $B_n=0$ if

$$(5)\quad \int_{-\infty}^{\infty}\frac{dw}{(\rho^2+w^2)^{n+1}}Q_n\left(\frac{w^2-\rho^2}{w^2+\rho^2}\right) = 0.$$

To establish this relation we make use of the expansion

$$(6)\quad \sum_{n=0}^{\infty}h^nQ_n(\nu) = (1-2\nu h+h^2)^{-1/2}\sinh^{-1}\left[(\nu-h)(1-\nu^2)^{-1/2}\right]$$
$$|h|<1,\quad (-1<\nu<1),$$

which is readily derived from the formula* defining $Q_n(\nu)$, namely,

$$(7)\quad \frac{1}{r^{n+1}}Q_n\left(\frac{z}{r}\right) = \frac{(-1)^n}{2^n\cdot n!}\frac{\partial^n}{\partial z^n}\left(\frac{1}{2r}\log\frac{r+z}{r-z}\right).$$

* E. W. Hobson, Proceedings of the London Mathematical Society, (1), vol. 22 (1891), p. 438. (This reference was given to me by G. N. Watson.)

The expansion indicates that $Q_n[(w^2-\rho^2)/(w^2+\rho^2)]$ can be regarded as an odd function of w and so the truth of (5) becomes manifest. To determine the value of A_n we shall first obtain another integral for the same Newtonian potential by making use of the fact that if the function $W=F(x, y, z^2-t^2)$ is a solution of the wave-equation

$$\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}+\frac{\partial^2 W}{\partial z^2}=\frac{\partial^2 W}{\partial t^2},$$

the integral

$$V=\frac{1}{2}\int_{t-z}^{t+z} F[x,y,z^2-(t-\tau)^2]d\tau$$

is in many cases a solution of Laplace's equation. Instead of enumerating a set of sufficient conditions to be satisfied by F we shall simply remark that the conditions (indicated by the differentiations under the integral sign) are evidently satisfied in the present case (when n is a positive integer) because the integrand in the integral now to be considered is a polynomial:

$$(8) \qquad \frac{1}{2}\int_{t-z}^{t+z} d\tau[r^2-(t-\tau)^2]^n P_n\left[\frac{\rho^2-z^2+(t-r)^2}{\rho^2+z^2-(t-r)^2}\right] = (-1)^n C_n r^{-2n-1} Q_{2n}(z/r)\ .$$

The integral evidently represents a function of type $r^{-2n-1}F(z/r)$ and is zero when $z=0$, consequently it represents a constant multiple of $r^{-2n-1}Q_{2n}(z/r)$. The constant multiplier may be determined by differentiating with respect to z and making use of the relation

$$Q'_{2n}(0)=(-1)^n/C_n.$$

This may be proved by differentiating the relation (5) with respect to z and then putting $z=0$ and using the expansion

$$(9)\ h(1+h^2)^{-3/2}\sinh^{-1} h-1/(1+h^2)=\sum_{n=0}^{\infty}(-1)^{n+1}h^n/C_n.$$

Multiplying the relation (4) by a^{2n} and summing from $n=0$ to $n=\infty$, we obtain the relation

$$(10) \qquad \int_0^{\phi}\frac{d\theta}{(r_1^2-r_2^2\sin^2\theta)^{1/2}}=\sum_{n=0}^{\infty}(-1)^n C_n(a^{2n}/r^{2n+1})Q_{2n}(z/r),$$

where $r_1^2=(\rho+a)^2+z^2$, $r_2^2=(\rho-a)^2+z^2$, and $r_2 \sin\phi = z$. This relation holds for $\rho \geqq a$.

Calling the potential function on the left of (10) v, we note that if (4) is correct, there must be a relation of type

$$v = \frac{1}{\pi}\int_{-\infty}^{\infty} dw\left[a^4 + 2a^2(z^2+w^2-\rho^2) + (r^2+w^2)^2\right]^{-1/2} \sinh^{-1}\left[\frac{a^2+z^2+w^2-\rho^2}{2\rho(z^2+w^2)^{1/2}}\right], \qquad \rho \geqq a.$$

To test this relation we first put $\rho = a$. It then becomes

$$\int_0^{\pi/2} \frac{d\theta}{(4a^2+z^2\cos^2\theta)^{1/2}} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dw}{[(z^2+w^2)(z^2+w^2+4a^2)]^{1/2}} \sinh^{-1}\left[\frac{(z^2+w^2)^{1/2}}{2a}\right].$$

This relation may be checked by expansion in powers of $1/a$, making use of the expansion

$$(11)\quad (z^2+w^2+4a^2)^{-1/2}\sinh^{-1}\left[\frac{(z^2+w^2)^{1/2}}{2a}\right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)C_n}\,\frac{(z^2+w^2)^{n+1/2}}{(2a)^{2n+2}}.$$

This will serve as an outline of a method by which the relation (4) may be established.

It should be noticed that by making use of (7) the series on the right of (10) may be expressed in the form

$$(12)\qquad v = J_0\left[a\frac{\partial}{\partial z}\right]\left\{\frac{1}{2r}\log\frac{r+z}{r-z}\right\},$$

and, since*

$$(13)\qquad \frac{1}{2r}\log\frac{r+z}{r-z} = -\frac{\pi}{2}\int_0^{\infty} e^{-kz}Y_0(k\rho)\,dk,$$

the last expression may be replaced by a definite integral. The appropriate formula is

* Watson, *Bessel Functions*, p. 387.

$$(14)\qquad \begin{cases} v = -\dfrac{1}{2}\pi\displaystyle\int_0^\infty e^{-kz}J_0(ka)Y_0(k\rho)dk, & (\rho > a > 0), \\ v = -\dfrac{1}{2}\pi\displaystyle\int_0^\infty e^{-kz}Y_0(ka)J_0(k\rho)dk, & (a > \rho > 0). \end{cases}$$

G. N. Watson has kindly mentioned that the function v is also expressible in the forms

$$(15)\qquad \begin{aligned} v &= -\frac{1}{2}\int_0^\infty dk\int_0^\pi e^{-kz}Y_0(kR)d\phi \\ &= -\frac{1}{2}\int_0^\pi d\phi\int_0^\infty e^{-kz}Y_0(kR)dk, \quad (R^2 = \rho^2 + a^2 - 2\rho a\cos\phi), \end{aligned}$$

the repeated integral being absolutely convergent.

Formula (14) is easily checked by noticing that v is a symmetric function of ρ and a and that a correct result is obtained by putting $\rho=0$. That the result is also correct for $\rho=a$ is seen by making use of the well known formula*

$$(16)\qquad J_0(ka)Y_0(ka) = -\frac{2}{\pi}\int_0^\infty J_0(2ka\cosh u)du.$$

The equation then becomes

$$\int_0^\infty (4a^2 + \cos^2\theta)^{-1/2}d\theta = \int_0^\infty (z^2 + 4a^2\cosh^2 u)^{-1/2}du.$$

If we first put $\rho=a$ in v and then make $z=0$ the result is $\pi/(4a)$ but if we first put $z=0$ and then make $\rho\rightarrow a$ the result is zero, a result which is in agreement with the fact that v is an odd function of z.

It is interesting to note that

$$(17)\qquad \begin{aligned} \int_0^\infty J_0(\lambda a)Y_0(\lambda\rho)d\lambda &= 0, & (\rho > a > 0), \\ &= -1/(2a), & (\rho = a > 0). \end{aligned}$$

The value of the integral for $\rho < a$ has been obtained for me by G. N. Watson. When $0<\rho<a$ it is

---

* N. Nielsen, *Handbuch der Theorie der Cylinder Funktionen*, p. 215.

$$-\frac{2}{\pi a}K'\left(\frac{\rho}{a}\right)$$

where K and iK' are the quarter periods of elliptic functions of modulus k. It should be observed that

$$\lim_{\rho\to a-0}\int_0^\infty J_0(\lambda a)Y_0(\lambda\rho)d\lambda = -1/a. \tag{18}$$

When $\rho<0$, $a>0$, $\rho=\sigma e^{i\pi}$ $(\sigma>0)$, $Y_0(\lambda\rho)=Y_0(\lambda\sigma)+2iJ_0(\lambda\sigma)$, the value is

$$-\frac{2}{\pi a}K'\left(\frac{\sigma}{a}\right)+\frac{4i}{\pi a}K\left(\frac{\sigma}{a}\right)$$

for $\sigma<a$ and $[4i/(\pi\sigma)]K(a/\sigma)$ for $\sigma>a$. When $\sigma=a$ the integral is divergent.

Watson remarks that a combination of (15) with (13) gives the simple formula

$$v=\frac{1}{2\pi}\int_0^\pi\frac{d\phi}{r'}\log\frac{r'+z}{r'-z},\qquad (r'^2=z^2+R^2). \tag{19}$$

Two other simple expressions for v may be obtained by using the known formula*

$$Y_0(x) = -\frac{2}{\pi}\int_0^\infty \cos(x\cosh u)du,\qquad (x>0). \tag{20}$$

The result is that

$$v = \text{real part of}\begin{cases}\displaystyle\int_0^\infty du[(z+i\rho\cosh u)^2+a^2]^{-1/2}, & (\rho\geqq a),\\ \displaystyle\int_0^\infty du[(z+ia\cosh u)^2+\rho^2]^{-1/2}, & (\rho\leqq a).\end{cases} \tag{21}$$

Here v is the potential corresponding to a surface density proportional to $1/(az)$ on the cylinder $\rho=a$. The discontinuity in the surface density at the plane $z=0$ is responsible for the discontinuity in v. We have actually

* When $\rho>a>0$ the result is easily deduced from Nielsen's formula (12) on p. 193 of his book.

$$(22)\quad \begin{cases} [\partial v/\partial \rho]_{\rho=a+0} - [\partial v/\partial \rho]_{\rho=a-0} = -1/(az), \\ [v(\rho,z)]_{\rho=a,z=+0} - [v(\rho,z)]_{\rho=a,z=-0} = \pi/(2a). \end{cases}$$

The behavior of the potential would be precisely the same if the surface density proportional to $1/(az)$ were distributed on the sphere $r=a$ instead of the cylinder $\rho=a$. The potential in this case can be expressed in the forms

$$(23)\quad \begin{cases} v_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)C_n} \frac{r^{2n+1}}{a^{2n+2}} P_{2n+1}(z/r), & (r \leqq a), \\ v_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)C_n} \frac{a^{2n+1}}{r^{2n+2}} P_{2n+1}(z/r), & (r \geqq a). \end{cases}$$

For $r \leqq a$ we have, indeed, $v_1 = v$. This may be seen by making use of the relation

$$(24)\quad \frac{1}{2}\int_{t-z}^{t+z} d\tau [r^2 - (t-\tau)^2]^n P_n\left[\frac{\rho^2 - z^2 + (t-\tau)^2}{\rho^2 + z^2 - (t-\tau)^2}\right] = \frac{(-1)^n r^{2n+1}}{(2n+1)C_n} P_{2n+1}\left(\frac{z}{r}\right),$$

which is proved* in the same way as (8) when use is made of the relation

$$(25)\quad P'_{2n+1}(0) = (-1)^n/[(2n+1)C_n].$$

California Institute of Technology

* Watson, *Bessel Functions*, p. 180.