A DEFINITION OF AN UNKNOTTED SIMPLE CLOSED CURVE*

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Mazurkiewicz and Strasziewicz have given a definition of interlacability.† That this definition can be altered to apply to a single curve so as to test for a knot in the curve will be shown in what follows.

Consider a three-dimensional, euclidean space. If for each point of the interval $t_1 \leqq t \leqq t_2$ we let the continuous function $f(t)$ define a point in the space, the set of points so formed will be a continuous curve. If $f(t_1) = f(t_2)$ the curve is a closed curve. If $f(t') = f(t'')$ implies that either $t' = t_1$, $t'' = t_2$ or $t' = t_2$, $t'' = t_1$ or $t' = t''$, then the curve is a simple closed curve and will be denoted by the symbol $[f(t); t_1, t_2]$.

DEFINITION. *If* $[f(t); t_1, t_2]$ *is a simple closed curve and if a uniformly continuous function* $f(t, \lambda)$, *where* $t_1 \leqq t \leqq t_2$, $0 \leqq \lambda \leqq 1$, *can be found having the following properties*:

$$(1) \qquad f(t,1) = f(t);$$

$$(2) \qquad f(t,0) = f_0, \quad \textit{a constant};$$

$$(3) \qquad f(t',\lambda') = f(t'',\lambda''),$$

if and only if (i) $\lambda' = \lambda'' = 0$, *or* (ii) $\lambda' = \lambda'' \neq 0$, *and one of the following hold*: (a) $t' = t''$, *or* (b) $t' = t_1$, $t'' = t_2$, *or* (c) $t' = t_2$, $t'' = t_1$; *then the simple closed curve* $[f(t); t_1, t_2]$ *is unknotted.*

It must now be shown that the ordinary properties of knots are impossible for curves satisfying this definition; that is, no knotted curve can be a subset of a set which is in 1–1 continuous correspondence with a plane and every unknotted curve can be exhibited as a subset of such a set.

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† Mazurkiewicz and Strasziewicz, *Sur les coupures de l'espace*, Fundamenta Mathematicae, vol. 9, p. 205.

THEOREM 1. *An unknotted simple closed curve can be exhibited as a subset of a set which is in* 1–1 *continuous correspondence with a plane.*

PROOF. Consider the set $[f(\bar{t},\lambda)]$, where $\bar{t}$ is a constant and λ varies. If $\bar{t}$, $t'(\neq \bar{t})$ are not both members of the pair of values t_1, t_2, this set $[f(\bar{t},\lambda)]$ has no point in common with $[f(t',\lambda)]$ except f_0. Also each of these sets is an arc from f_0 to a point of $[f(t); t_1, t_2]$ since their points are in 1–1 continuous correspondence with the points of the segment $0 \leqq \lambda \leqq 1$, and $f(\bar{t}, 0) = f(t', 0) = f_0$.

From the fact that $f(t, \lambda)$ is uniformly continuous, it is necessary that the system of arcs $[f(t,\lambda)]$, $t_1 \leqq t \leqq t_2$, be self-compact. Moreover the arcs $[f(t_1, \lambda)]$ and $[f(t_2, \lambda)]$ are identical.

It will now be shown that the system of arcs is an equicontinuous system.* Suppose they do not form such a system. This means that there exists an ϵ such that no matter how small $\delta_{\epsilon 1}$ we choose, it must be possible to find an arc of the system which contains two points $f(t_1', \lambda_{11})$, $f(t_1', \lambda_{12})$ such that these points are at a distance $<\delta_{\epsilon 1}$ apart, but there exists a value λ_{13} where $\lambda_{11} < \lambda_{13} < \lambda_{12}$, and the distance from $f(t_1', \lambda_{13})$ to $f(t_1', \lambda_{11})$ is $>\epsilon$. For the arc $[f(t_1', \lambda)]$, however, there exists a δ_1 such that for all values of $\delta < \delta_1$ every pair of points at distance $\leqq \delta$ apart can be joined by an arc of length $<\epsilon$, which joining arc is a subarc of the original. Hence if we choose $\delta_{\epsilon 2}$ less than the smaller of the two δ_1, $\delta_{\epsilon 1}$, we can be sure that there is an arc $[f(t_2', \lambda)]$ distinct from $[f(t_1', \lambda)]$ for which there are two points $f(t_2', \lambda_{21})$, $f(t_2', \lambda_{22})$ at a distance $<\delta_{\epsilon 2}$ apart but on which there also exists a point $f(t_2', \lambda_{23})$ such that $\lambda_{21} < \lambda_{23} < \lambda_{22}$ and the distance from $f(t_2', \lambda_{23})$ to $f(t_2', \lambda_{21})$ is greater than ϵ. By a proper choice of $\delta_{\epsilon 3}$ we proceed with this process and since the set of arcs was supposed not equicontinuous we must arrive at an infinite set of arcs. Since all values of t lie between t_1 and t_2, the sequence $t_1', t_2', t_3', \cdots$ must have a limit T. As we can discard members of this sequence until T is the only limit, it will be supposed that this has been done and that T is the sequential limit of the sequence $t_1', \cdots$. In the same way it may be supposed that the sequence $\lambda_{1i}, \lambda_{2i}, \lambda_{3i}, \cdots$ must have

* For definition see R. L. Moore, *On the generation of a surface*, Fundamenta Mathematicae, vol. 4, p. 106, footnote 3.

a unique limit Λ_i for $i=1, 2, 3$. Now on the arc $[f(T, \lambda)]$ we can make the point $f(T, \Lambda_1)$ the same as the point $f(T, \Lambda_2)$ because the sequence $\delta_{\epsilon 1}, \delta_{\epsilon 2}, \delta_{\epsilon 3}, \cdots$ can be chosen to converge to zero and still satisfy all other conditions so far imposed on it, and then $|\lambda_{j1}-\lambda_{j2}|\rightarrow 0$ as $j\rightarrow\infty$. But since we choose $\lambda_{j1}<\lambda_{j3}<\lambda_{j2}$ we must also have $\Lambda_3=\Lambda_1$. This, however, contradicts the fact that the distance from $f(t_j', \lambda_{j1})$ to $f(t_j', \lambda_{j2})$ was chosen greater than ϵ for every j, because this choice means that the smallest possible distance from $f(T, \Lambda_1)$ to $f(T, \Lambda_3)$ is ϵ. This shows that the assumption that the arcs do not form an equicontinuous system is false.

By a theorem due to R. L. Moore* this set of arcs $[f(t, \lambda)]$ is in 1–1 continuous correspondence with a plane set composed of a circle plus its interior. From this the theorem follows.

THEOREM 2. *Any simple closed curve M in a set S, which set S is in* 1–1 *continuous correspondence with a plane set S' consisting of a circle plus its interior, must be unknotted.*

PROOF. Let S' be a circle C of radius one about the point O, together with its interior. Let M' denote the set of points in S' that correspond to M under the given correspondence. Let the set of points of S that correspond to the interior of M' in the plane be called the interior of M. By a theorem due to Schönflies it is possible to make M plus its interior correspond to C plus its interior.† Then the straight line segment from O to a point of C will correspond to an arc from f_0 (which can be made the correspondent of O) to that point of M which is the correspondent of the point of C on the straight line segment. These segments enable us to set up a polar coordinate system on the set S'. If f is the function which gives the correspondence between S and S', we see that if a point (θ, r) is given on S', $f(\theta, r)$ will be its correspondent on S. Since $f(0, r)=f(2\pi, r)$ and the other properties of the function used in the definition of an unknotted curve are satisfied if $0\leqq\theta\leqq 2\pi$ and $0\leqq r\leqq 1$, M must be unknotted.

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* R. L. Moore, *On the generation of a surface*, loc. cit., p. 117, Theorem 3.

† See J. R. Kline, Proceedings of the National Academy of Sciences, vol. 6 (1920), p. 530, for a statement of this theorem.