NOTE ON PERIODIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES*

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Let $f_1(x), f_2(x), \cdots, f_m(x)$ be periodic functions of the complex variable $x$, each meromorphic in its fundamental domain of periodicity $\mathcal{G}$ (parallelogram or closed strip of periods). Let each function admit the period or periods corresponding to $\mathcal{G}$, and let there be no other periods common to all the functions except such as are derived linearly and integrally from those of $\mathcal{G}$. Then a suitable linear combination of the above functions,

$$C_1f_1(x) + \cdots + C_mf_m(x),$$

will admit the periods corresponding to $\mathcal{G}$, and no others.

The corresponding theorem is not true for periodic functions of several complex variables. The proof is given by the following example. Let

$$F(u_1, u_2, u_3) = u_1 - \zeta(u_2),$$
$$\Phi(u_1, u_2, u_3) = u_2 - \zeta(u_3),$$

where

$$\zeta(z) = \frac{d}{dz} \log \sigma(z) = \frac{\sigma'(z)}{\sigma(z)},$$

and $\sigma(z)$ is the Weierstrassian sigma-function. Here

$$\zeta(z + \omega_1) = \zeta(z) + \eta_1,$$
$$\zeta(z + \omega_2) = \zeta(z) + \eta_2,$$

where $\omega_1, \omega_2, \eta_1, \eta_2$ are connected by Legendre's relation,

$$\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i.$$

These functions $F$ and $\Phi$ obviously admit the periods

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Moreover, these (and their integral combinations) are the only periods. For, let \((P_1, P_2, P_3)\) be an arbitrary period. Then
\[
\begin{align*}
\psi_1 + P_1 - \zeta(u_3 + P_3) &= \psi_1 - \zeta(u_3), \\
\psi_2 + P_2 - \zeta(u_3 + P_3) &= \psi_2 - \zeta(u_3).
\end{align*}
\]
Hence \(P_1 = P_2\) and
\[
\zeta(u_3 + P_3) - \zeta(u_3) = P_1.
\]

In order that the function on the left-hand side of this identity admit no poles, it is necessary and sufficient that
\[
P_3 = m_1 \psi_1 + m_2 \psi_2,
\]
where \(m_1, m_2\) are whole numbers. But then
\[
P_1 = P_2 = m_1 \psi_1 + m_2 \psi_2.
\]

Consider now an arbitrary linear combination of these functions. Such a function has the value
\[
AF(u_1, u_2, u_3) + B\Phi(u_1, u_2, u_3) = (Au_1 + Bu_2) - (A + B)\zeta(u_3).
\]
It is seen to depend on fewer than three linear combinations of \(u_1, u_2, u_3\), for if we set
\[
w_1 = Au_1 + Bu_2, \quad w_2 = u_3,
\]
the function becomes
\[
w_1 + (A + B)\zeta(w_2).
\]
Hence the function \(AF + B\Phi\) admits infinitely small periods, and the proof is complete.

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